On the second mixed moment of the characteristic polynomials of the 1D band matrices

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Abstract

We consider the asymptotic behavior of the second mixed moment of the characteristic polynomials of the 1D Gaussian band matrices, i.e. of the hermitian matrices H_n with independent Gaussian entries such that $\langle H_{ij}H_{lk}\rangle = \delta_{ik}\delta_{jl}J_{ij}$, where $J = (-W^2\triangle + 1)^{-1}$. Assuming that $W^2 = n^{1+\theta}$, $0 < \theta < 1$, we show that this asymptotic behavior (as $n \to \infty$) in the bulk of the spectrum coincides with those for the Gaussian Unitary Ensemble.

1 Introduction

The hermitian Gaussian 1D random band matrices (RBM) are hermitian $(2n+1)\times(2n+1)$ matrices H_n (we numerate indices of entries from $-n$ to n) whose entries H_{jk} are random Gaussian variables with mean zero such that

$$\langle H_{ij}H_{lk}\rangle = \delta_{ik}\delta_{jl}J_{ij}, \tag{1.1}$$

where J_{ij} is a symmetric function which is small for large $|i-j|$ and

$$\sum_{i=-n}^{n} J_{ij} = 1.$$

In this paper we consider the especially convenient choice of J_{ij}, which is given by the lattice Green's function

$$J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij}, \tag{1.2}$$

where Δ is the discrete Laplacian on $[-n, n]$, i.e.

$$(-\Delta f)_j = \begin{cases} -f_{j-1} + 2f_j - f_{j+1}, & j \neq -n, n, \\ -f_{j-1} + f_j - f_{j+1}, & j = -n, n. \end{cases} \tag{1.3}$$

Note that for this choice of J we have $J_{ij} \approx W^{-1}\exp\{-|i-j|/W\}$, and so the variance of the matrix elements is exponentially small when $|i-j| \gg W$. Hence W can be considered as the width of the band.

The probability law of the RBM H_n can be written in the form

$$P_n(dH_n) = \prod_{-n\le i<j\le n} \frac{dH_{ij}dH_{ij}^*}{2\pi J_{ij}} e^{-\frac{|H_{ij}|^2}{J_{ij}}} \prod_{i=-n}^{n} \frac{dH_{ii}}{\sqrt{2\pi J_{ii}}} e^{-\frac{H_{ii}^2}{2J_{ii}}}, \tag{1.4}$$

We assume also that $W^2 = n^{1+\theta}$, $0 < \theta < 1$, i.e. the width of the band tends to infinity faster than $\sqrt{n}$, when $n \to \infty$.

Let $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of H_n. Define their Normalized Counting Measure (NCM) as

$$N_n(\triangle) = \sharp\{\lambda_j^{(n)} \in \triangle, j = 1, \dots, n\}/n, \quad N_n(\mathbb{R}) = 1, \tag{1.5}$$

where $\triangle$ is an arbitrary interval of the real axis. For the RBM it was shown in [15] that N_n converges weakly to a non-random measure N which is called the limiting NCM of the ensemble. The measure N is absolutely continuous and its density ρ is given by the well-known Wigner semicircle law:

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad \lambda \in [-2, 2]. \tag{1.6}$$

Characteristic polynomials of random matrices have been actively studied in the last years (see e.g. [1, 4, 5, 7, 9, 10, 12, 14, 16, 17, 18, 19]). The interest to this topic is stimulated by its connections to the number theory, quantum chaos, integrable systems, combinatorics, representation theory and others.

The second correlation function (or the second mixed moment) of the characteristic polynomials is

$$F_2(\Lambda) = \int \det(\lambda_1 - H_n)\det(\lambda_2 - H_n) P_n(d\,H_n), \tag{1.7}$$

where $P_n(d\,H_n)$ is defined in (1.4), and $\Lambda = \mathrm{diag}\,\{\lambda_1, \lambda_2\}$ are real or complex parameters that may depend on n.

We are interested in the asymptotic behavior of (1.7) for matrices (1.1) – (1.4) as $n, W \to \infty$ $W^2 = n^{1+\theta}$, $0 < \theta < 1$, and for

$$\lambda_j = \lambda_0 + \frac{\xi_j}{n\rho(\lambda_0)}, \quad j = 1, 2,$$

where $\lambda_0 \in (-2, 2)$, ρ is defined in (1.6), and $\widehat{\xi} = \mathrm{diag}\,\{\xi_1, \xi_2\}$ are real parameters varying in the interval $[-M, M] \subset \mathbb{R}$.

Set also

$$D_2 = \prod_{l=1}^{2} F_2^{1/2}\Big(\lambda_0 + \frac{\xi_l}{n\rho(\lambda_0)}, \lambda_0 + \frac{\xi_l}{n\rho(\lambda_0)}\Big). \tag{1.8}$$

The main result of the paper is the following theorem:

Theorem 1. *Consider the random matrices (1.1) – (1.4). Define the second mixed moment F_2 of the characteristic polynomials as in (1.7). Then we have*

$$\lim_{n\to\infty} D_2^{-1} F_2\Big(\Lambda_0 + \widehat{\xi}/(n\rho(\lambda_0))\Big) = \frac{\sin(\pi(\xi_1 - \xi_2))}{\pi(\xi_1 - \xi_2)}, \tag{1.9}$$

where $\rho(\lambda)$ and D_2 are defined in (1.6) and (1.8), $\Lambda_0 = \mathrm{diag}\,\{\lambda_0, \lambda_0\}$, $\lambda_0 \in (-2, 2)$, $\widehat{\xi} = \mathrm{diag}\,\{\xi_1, \xi_2\}$, $\xi_1, \xi_2 \in [-M, M]$.

The theorem shows that the above limits for the second mixed moment of the characteristic polynomials for the 1D Gaussian random band matrices (with $W^2 = n^{1+\theta}$, $0 < \theta < 1$) coincide with those for the Gaussian Unitary Ensemble, i.e. the local behavior of the second mixed moment in the bulk of the spectrum is universal.

The paper is organized as follows. In Section 2 we obtain a convenient asymptotic integral representation for F_2, using the integration over the Grassmann variables and the Harish Chandra/Itzykson-Zuber formula for integrals over the unitary group. The method is a generalization of that of [3, 4] and is an analog of the method of [16, 17], where the hermitian Wigner and general sample covariance matrices were considered. In Section 3 we prove Theorem 1, applying the steepest descent method to the integral representation. Appendix is devoted to the proofs of the auxiliary statements.

We denote by C, C_1, K, etc. various n-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis.

2 Integral representation

In this section we obtain the integral representation for F_2 of (1.7) by using integration over the Grassmann variables. This method allows us to obtain the integral representation of the product of the characteristic polynomials which is very useful for the averaging because it looks like the Gaussian-type integral (see the formula (2.7) below). After averaging over the probability measure we can integrate over the Grassmann variables to obtain the usual asymptotic integral representation which can be studied by the steepest descent method.

The integration over the Grassmann variables was introduced by Berezin and widely used in the physics literature (see e.g. [2] and [6]). For the reader convenience we give a brief outline of the techniques.

2.1 Grassmann integration

Let us consider two sets of formal variables $\{\psi_j\}_{j=1}^n$, $\{\overline{\psi}_j\}_{j=1}^n$, which satisfy the anticommutation conditions

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0, \quad j,k = 1,\dots,n. \tag{2.1}$$

These two sets of variables $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$ generate the Grassmann algebra $\mathfrak{A}$. Taking into account that $\psi_j^2 = 0$, we have that all elements of $\mathfrak{A}$ are polynomials of $\{\psi_j\}_{j=1}^n$ and $\{\overline{\psi}_j\}_{j=1}^n$. We can also define functions of the Grassmann variables. Let χ be an element of $\mathfrak{A}$, i.e.

$$\chi = a + \sum_{j=1}^n (a_j\psi_j + b_j\overline{\psi}_j) + \sum_{j\neq k}(a_{j,k}\psi_j\psi_k + b_{j,k}\psi_j\overline{\psi}_k + c_{j,k}\overline{\psi}_j\overline{\psi}_k) + \dots. \tag{2.2}$$

For any analytical function f we mean by $f(\chi)$ the element of $\mathfrak{A}$ obtained by substituting $\chi - a$ in the Taylor series of f at the point a. Since χ is a polynomial of $\{\psi_j\}_{j=1}^n$, $\{\overline{\psi}_j\}_{j=1}^n$

of the form (2.2), according to (2.1) there exists such l that $(\chi - a)^l = 0$, and hence the series terminates after a finite number of terms and so $f(\chi) \in \mathfrak{A}$.

For example, we have

$$\begin{aligned}
&\exp\{b\overline{\psi}_j\psi_j\} = 1 + b\overline{\psi}_j\psi_j + (b\overline{\psi}_j\psi_j)^2/2 = 1 + b\overline{\psi}_j\psi_j, \\
&\exp\{a_{11}\overline{\psi}_1\psi_1 + a_{12}\overline{\psi}_1\psi_2 + a_{21}\overline{\psi}_2\psi_1 + a_{22}\overline{\psi}_2\psi_2\} = 1 + a_{11}\overline{\psi}_1\psi_1 \\
&+ a_{12}\overline{\psi}_1\psi_2 + a_{21}\overline{\psi}_2\psi_1 + a_{22}\overline{\psi}_2\psi_2 + (a_{11}\overline{\psi}_1\psi_1 + a_{12}\overline{\psi}_1\psi_2 \\
&+ a_{21}\overline{\psi}_2\psi_1 + a_{22}\overline{\psi}_2\psi_2)^2/2 = 1 + a_{11}\overline{\psi}_1\psi_1 + a_{12}\overline{\psi}_1\psi_2 + a_{21}\overline{\psi}_2\psi_1 \\
&+ a_{22}\overline{\psi}_2\psi_2 + (a_{11}a_{22} - a_{12}a_{21})\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2.
\end{aligned} \tag{2.3}$$

Following Berezin [2], we define the operation of integration with respect to the anticommuting variables in a formal way:

$$\int d\,\psi_j = \int d\,\overline{\psi}_j = 0, \quad \int \psi_j d\,\psi_j = \int \overline{\psi}_j d\,\overline{\psi}_j = 1. \tag{2.4}$$

This definition can be extended on the general element of $\mathfrak{A}$ by the linearity. A multiple integral is defined to be a repeated integral. The "differentials" $d\,\psi_j$ and $d\,\overline{\psi}_k$ anticommute with each other and with the variables ψ_j and $\overline{\psi}_k$.

Thus, if

$$f(\eta_1, \ldots, \eta_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1}\eta_{j_1} + \sum_{j_1<j_2} p_{j_1 j_2}\eta_{j_1}\eta_{j_2} + \ldots + p_{1,2,\ldots,k}\eta_1 \ldots \eta_k,$$

where $\eta_1, \ldots, \eta_k$ are some elements from the sets $\{\psi_j\}_{j=1}^n$, $\{\overline{\psi}_j\}_{j=1}^n$, then

$$\int f(\eta_1, \ldots, \eta_k) d\,\eta_k \ldots d\,\eta_1 = p_{1,2,\ldots,k}. \tag{2.5}$$

Let A be an ordinary hermitian matrix. The following Gaussian integral is well-known

$$\int \exp\Big\{ - \sum_{j,k=1}^{n} A_{j,k} z_j \overline{z}_k \Big\} \prod_{j=1}^{n} \frac{d\,\Re z_j d\,\Im z_j}{\pi} = \frac{1}{\det A}. \tag{2.6}$$

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see [2]):

$$\int \exp\Big\{ - \sum_{j,k=1}^{n} A_{j,k} \overline{\psi}_j \psi_k \Big\} \prod_{j=1}^{n} d\,\overline{\psi}_j d\,\psi_j = \det A. \tag{2.7}$$

For $n = 1$ and $n = 2$ this formula follows immediately from (2.3) and (2.5).

Besides, we have

$$\int \prod_{p=1}^{q} \overline{\psi}_{l_p}\psi_{s_p} \exp\Big\{ - \sum_{j,k=1}^{n} A_{j,k} \overline{\psi}_j \psi_k \Big\} \prod_{j=1}^{n} d\,\overline{\psi}_j d\,\psi_j = \det A_{l_1,..,l_q;s_1,..,s_q}, \tag{2.8}$$

where $A_{l_1,..,l_q;s_1,..,s_q}$ is a $(n-q)\times(n-q)$ minor of the matrix A without lines $l_1, \ldots, l_q$ and columns $s_1, \ldots, s_q$.

2.2 Formula for F_2

Using (2.7) we obtain

$$
\begin{aligned}
F_2(\Lambda) &= \mathbf{E}\Big\{ \int e^{-\sum\limits_{l=1}^{2}\sum\limits_{j,k=-n}^{n}(\lambda_l - H_n)_{j,k}\overline{\psi}_{jl}\psi_{kl}} \prod_{r=1}^{2}\prod_{q=-n}^{n} d\,\overline{\psi}_{qr} d\,\psi_{qr} \Big\} \\
&= \mathbf{E}\Big\{ \int e^{-\sum\limits_{s=1}^{2}\lambda_s \sum\limits_{p=-n}^{n}\overline{\psi}_{ps}\psi_{ps}} \exp\Big\{ \sum_{j<k}\sum_{l=1}^{2} \Big(\Re H_{j,k}\cdot(\overline{\psi}_{jl}\psi_{kl} + \overline{\psi}_{kl}\psi_{jl}) \qquad (2.9) \\
&\quad + i\Im H_{j,k}\cdot(\overline{\psi}_{jl}\psi_{kl} - \overline{\psi}_{kl}\psi_{jl}) \Big) + \sum_{j=-n}^{n} H_{jj}\cdot \sum_{l=1}^{2}\overline{\psi}_{jl}\psi_{jl} \Big\} \prod_{r=1}^{2}\prod_{q=-n}^{n} d\,\overline{\psi}_{qr} d\,\psi_{qr} \Big\},
\end{aligned}
$$

where $\{\psi_{jl}\}$, $j = -n, \ldots, n$, $l = 1, 2$ are the Grassmann variables ($2n+1$ variables for each determinant in (1.7)), and $\mathbf{E}\{\ldots\}$ is an expectation with respect to the measure (1.4). Integrating over the measure (1.4) we get

$$
F_2(\Lambda) = \int \prod_{r=1}^{2}\prod_{q=-n}^{n} d\,\overline{\psi}_{qr} d\,\psi_{qr} \exp\Big\{ -\sum_{s=1}^{2}\lambda_s \sum_{p=1}^{n}\overline{\psi}_{ps}\psi_{ps} \Big\} \qquad (2.10)
$$
$$
\times \exp\Big\{ \sum_{j<k} J_{j,k}(\overline{\psi}_{j1}\psi_{k1} + \overline{\psi}_{j2}\psi_{k2})(\overline{\psi}_{k1}\psi_{j1} + \overline{\psi}_{k2}\psi_{j2}) + \sum_{j=-n}^{n} \frac{J_{j,j}}{2}(\overline{\psi}_{j1}\psi_{j1} + \overline{\psi}_{j2}\psi_{j2})^2 \Big\}.
$$

Use the well-known Hubbard-Stratonovich transform:

$$
\begin{aligned}
&\int \exp\Big\{ -\frac{1}{2}\sum_{j,k} J^{-1}_{j,k} \mathrm{Tr}\, X_j X_k - i\sum_j (\overline{\psi}_{j1}, \overline{\psi}_{j2}) X_j \begin{pmatrix} \psi_{j1} \\ \psi_{j2} \end{pmatrix} \Big\} \prod_{j=-n}^{n} dX_j \\
&\qquad = (2\pi^2)^{2n+1}\det{}^2 J \cdot \exp\Big\{ \frac{1}{2}\sum_{j,k} J_{j,k}(\overline{\psi}_{j1}\psi_{k1} + \overline{\psi}_{j2}\psi_{k2})(\overline{\psi}_{k1}\psi_{j1} + \overline{\psi}_{k2}\psi_{j2}) \Big\},
\end{aligned}
$$

where X_j is hermitian 2×2 matrix and

$$
dX_j = d\Re X_{12}\, d\Im X_{12}\, dX_{11}\, dX_{22}. \qquad (2.11)
$$

Substituting this and expression (1.2) for J^{-1}_{jk} to (2.10) and using (2.7) to integrate over the Grassmann variables we obtain

$$
\begin{aligned}
F_2(\hat{\xi}) &= -(2\pi^2)^{-2n-1}\det{}^{-2} J \int_{\mathcal{H}_2} \exp\Big\{ -\frac{W^2}{2}\sum_{j=-n+1}^{n} \mathrm{Tr}\,(X_j - X_{j-1})^2 - \frac{1}{2}\sum_{j=-n}^{n}\mathrm{Tr}\, X_j^2 \Big\} \\
&\quad \times \prod_{j=-n}^{n} \det\big(X_j - i\Lambda_0 - i\hat{\xi}/n\rho(\lambda_0)\big) \prod_{j=-n}^{n} dX_j \qquad (2.12) \\
&= -(2\pi^2)^{-2n-1}\det{}^{-2} J \int_{\mathcal{H}_2} \exp\Big\{ -\frac{W^2}{2}\sum_{j=-n+1}^{n} \mathrm{Tr}\,(X_j - X_{j-1})^2 \Big\} \\
&\quad \times \exp\Big\{ -\frac{1}{2}\sum_{j=-n}^{n} \mathrm{Tr}\Big(X_j + \frac{i\Lambda_0}{2} + \frac{i\hat{\xi}}{n\rho(\lambda_0)} \Big)^2 \Big\} \prod_{j=-n}^{n} \det\big(X_j - i\Lambda_0/2\big) \prod_{j=-n}^{n} dX_j.
\end{aligned}
$$

Let us change the variables to $X_j = U_j^* A_j U_j$, where U_j is a unitary matrix and $A_j = \mathrm{diag}\,\{a_j, b_j\}$, $j = -n, \dots, n$. Then dX_j of (2.11) becomes (see e.g. [13], Section 3.3)

$$\frac{\pi}{2}(a_j - b_j)^2 da_j\, db_j d\mu(U_j),$$

where $d\mu(U_j)$ is the normalized to unity Haar measure on the unitary group $U(2)$. Thus we get

$$F_2\Big(\Lambda_0 + \frac{\hat{\xi}}{n\rho(\lambda_0)}\Big) = -\frac{\det^{-2} J}{(4\pi)^{2n+1}} \int \exp\Big\{ -\frac{W^2}{2} \sum_{j=-n+1}^{n} \mathrm{Tr}\,(U_j^* A_j U_j - U_{j-1}^* A_{j-1} U_{j-1})^2 \Big\}$$
$$\times \exp\Big\{ -\frac{1}{2} \sum_{j=-n}^{n} \mathrm{Tr}\,\Big(A_j + \frac{i\Lambda_0}{2}\Big)^2 + \frac{i}{n\rho(\lambda_0)} \sum_{j=-n}^{n} \mathrm{Tr}\, U_j^* A_j U_j \hat{\xi} \Big\}$$
$$\times \prod_{k=-n}^{n} (a_k - i\lambda_0/2)(b_k - i\lambda_0/2) \prod_{l=-n}^{n} (a_l - b_l)^2\, d\bar{a}\, d\bar{b}\, d\mu(U_q),$$

where

$$d\bar{a} = \prod_{j=-n}^{n} da_j, \quad d\bar{b} = \prod_{j=-n}^{n} db_j. \tag{2.13}$$

Note that we can change "angle variables" U_j to $V_j = U_j U_{j-1}^*$, $j = -n+1, \dots, n$ (i.e. the new variables are $U_{-n}, V_{-n+1}, V_{-n+2}, \dots, V_n$). Then we have

$$F_2\Big(\Lambda_0 + \frac{\hat{\xi}}{n\rho(\lambda_0)}\Big) = -\frac{\det^{-2} J}{(4\pi)^{2n+1}} \int_{U(2)}\!\!\int \exp\Big\{ -\frac{W^2}{2} \sum_{j=-n+1}^{n} \mathrm{Tr}\,(V_j^* A_j V_j - A_{j-1})^2 \Big\}$$
$$\times \exp\Big\{ -\tfrac{1}{2} \sum_{j=-n}^{n} \mathrm{Tr}\,\big(A_j + \tfrac{i\Lambda_0}{2}\big)^2 + \tfrac{i}{n\rho(\lambda_0)} \sum_{j=-n}^{n} \mathrm{Tr}\,\big(U_{-n} P_j\big)^* A_j\,(U_{-n} P_j)\hat{\xi} \Big\}$$
$$\times \prod_{k=-n}^{n} \Big((a_k - b_k)^2 (a_k - i\lambda_0/2)(b_k - i\lambda_0/2)\Big) d\mu(U_{-n})\, d\bar{a}\, d\bar{b} \prod_{p=-n+1}^{n} d\mu(V_q), \tag{2.14}$$

where

$$P_k = \prod_{s=-n+1}^{k} V_s. \tag{2.15}$$

3 Saddle-point analysis

3.1 Sketch of the proof

We can rewrite (2.14) as

$$F_2\Big(\Lambda_0 + \frac{\hat{\xi}}{n\rho(\lambda_0)}\Big) = -\frac{\det^{-2} J}{(4\pi)^{2n+1}} \int \exp\Big\{ -\frac{W^2}{2} \sum_{j=-n+1}^{n} \mathrm{Tr}\,(V_j^* A_j V_j - A_{j-1})^2 \Big\}$$
$$\times \exp\Big\{ -\sum_{j=-n}^{n} (f(a_j) + f(b_j)) + \tfrac{i}{n\rho(\lambda_0)} \sum_{j=-n}^{n} \mathrm{Tr}\,\big(U_{-n} P_j\big)^* A_j\,(U_{-n} P_j)\hat{\xi} \Big\}$$
$$\times \prod_{l=-n}^{n} (a_l - b_l)^2 d\,\mu(U_{-n})\, d\bar{a}\, d\bar{b} \prod_{q=-n+1}^{n} d\mu(V_q),$$

where

$$f(x) = (x + i\lambda_0/2)^2/2 - \log(x - i\lambda_0/2). \tag{3.1}$$

Note that

$$\frac{d}{dx}f(x) = x + i\lambda_0/2 - \frac{1}{x - i\lambda_0/2},$$

Hence, the expected saddle-points are

$$a_\pm = \pm\frac{\sqrt{4-\lambda_0^2}}{2} = \pm\pi\rho(\lambda_0). \tag{3.2}$$

Let Σ be the integral of (2.14) over the domain Ω_δ which is the union of δ-neighborhoods of the points $(\overline{a}_+,\overline{a}_+)$, $(\overline{a}_+,\overline{a}_-)$, $(\overline{a}_-,\overline{a}_+)$, $(\overline{a}_-,\overline{a}_-)$, where $\overline{a}_\pm = (a_\pm,\dots,a_\pm) \in \mathbb{R}^{2n+1}$, and let Σ_c be the integral over the complement of Ω_δ.

We are going to prove that

$$|\Sigma_c| \le Ce^{-W^{1-\varepsilon}}I_0, \tag{3.3}$$

where ε is sufficiently small and

$$I_0 \sim C_0 W^{-8n-2}(2\pi)^{2n+1}2^{2n}e^{2(2n+1)c_0}\det{}^{-1}\left(-\Delta + \frac{2\gamma_0}{W^2}\right), \tag{3.4}$$

where γ_0 is a real positive constant, and $c_0 = 1/2 - \lambda_0^2/4$.

Since below we will prove that

$$\Sigma \sim C_1 e^{O(n/W)}I_0, \tag{3.5}$$

we obtain then (recall that $W^2 = n^{1+\theta}$)

$$\frac{|\Sigma_c|}{|\Sigma + \Sigma_c|} \le C\frac{e^{-W^{1-\varepsilon}}I_0}{|e^{-W^{1-\varepsilon}} - C_1e^{O(n/W)}|I_0} \le C\,e^{-W^{1-\varepsilon}/2},$$

and thus

$$F_2\Big(\Lambda_0 + \frac{\hat{\xi}}{n\rho(\lambda_0)}\Big) = -\frac{\det^{-1}J}{(4\pi)^{2n+1}}\Sigma\,(1+o(1)).$$

As we will prove below, we can take $\delta = W^{-\kappa}$ with sufficiently small κ.

The next step is the calculation of Σ. We take the $W^{-\kappa}$-neighborhood of the one of the points $(\overline{a}_+,\overline{a}_+)$, $(\overline{a}_+,\overline{a}_-)$, $(\overline{a}_-,\overline{a}_+)$, $(\overline{a}_-,\overline{a}_-)$ (for example $(\overline{a}_+,\overline{a}_-)$). The idea is to expand

$$f(x) = c_\pm(x-a_\pm)^2 + s_3(x-a_\pm)^3 + \ldots = c_\pm(x-a_\pm)^2 + \varphi_\pm(x-a_\pm),$$
$$c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2}\cdot\sqrt{1-\frac{\lambda_0^2}{4}},$$

then to leave only the quadratic form in the exponent in (2.14), and then integrate over the unitary groups. After the integration and changing of the variables $a_j \to a_\pm + \widetilde{a}_j/W$, $b_j \to a_\pm + \widetilde{b}_j/W$, we will face with a problem to study a complex value Gaussian $(2n+1)$-dimensional distribution

$$\mu_\gamma(x) = \exp\Big\{-\frac{1}{2}\sum_{j=-n+1}^{n}(x_j - x_{j-1})^2 - \frac{\gamma}{W^2}\sum_{j=-n}^{n}x_j^2\Big\} \tag{3.6}$$

with $\gamma \in \mathbb{C}$, $\Re\gamma > 0$ (in our case $\gamma = c_+$ or c_-). We take $\delta = W^{-\kappa}$ and define

$$\langle \dots \rangle_0 = Z_{\delta,\gamma}^{-1} \int_{-\delta W}^{\delta W} (\dots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \qquad Z_{\delta,\gamma} = \int_{-\delta W}^{\delta W} \mu_\gamma(x) \prod_{q=-n}^{n} dx_q \tag{3.7}$$

$$\langle \dots \rangle = Z_{\gamma}^{-1} \int (\dots) \cdot \mu_\gamma(x) \prod_{q=-n}^{n} dx_q, \qquad Z_{\gamma} = \int \mu_\gamma(x) \prod_{q=-n}^{n} dx_q.$$

Moreover, let $\langle \dots \rangle_*$ (and also $\langle \dots \rangle_{0,*}$) be (3.7) with $\mu_{\Re\gamma}(x)$ instead of $\mu_\gamma(x)$.

To leave only the quadratic form in the exponent in (2.14), we have to prove that

$$\Big\langle \exp\Big\{ \sum_{j=-n}^{n} \varphi_\pm(x_j/W) \Big\} - 1 \Big\rangle_0 = o(1). \tag{3.8}$$

This can be done by using three ideas: (1) we can replace the integral over the neighborhood by the integral over all real axis with an error which we can control; (2) using the Wick's theorem we can prove that

$$|\langle x_{i_1}^{k_1} \dots x_{i_l}^{k_l} \rangle| \le \langle x_{i_1}^{k_1} \dots x_{i_l}^{k_l} \rangle_* \tag{3.9}$$

and thus we can estimate the averaging of some function over the complex measure by the averaging of the "changed" function over the positive one (we take $\Re c_\pm$ instead of $c_\pm$); (3) using the Wick's theorem we can prove (3.8) for the positive measure.

These ideas also help to prove that the main contribution to the integral of

$$\exp\Big\{ \frac{i}{n\rho(\lambda_0)} \mathrm{Tr}\, \big(U_{-n} \prod_{s=-n+1}^{j} V_s\big)^* (L + \widetilde{A}_s/W)\, (U_{-n} \prod_{s=-n+1}^{j} V_s) \hat{\xi} \Big\}$$

gives the term

$$\exp\Big\{ \frac{i}{n\rho(\lambda_0)} \mathrm{Tr}\, \big(U_{-n} \prod_{s=-n+1}^{j} V_s\big)^* L\, (U_{-n} \prod_{s=-n+1}^{j} V_s) \hat{\xi} \Big\}.$$

Here $L = \mathrm{diag}\{a_+, a_-\}$, $A_s = L + \widetilde{A}_s/W$.

The last step is to prove that in the functions above we can replace all V_s by I_2.

3.2 Proof of (3.3)

According to (2.14) we have

$$\begin{aligned} |\Sigma_c| \le \int_{\Omega_\delta^C} & \exp\Big\{ -\frac{1}{2}\Re \sum_{j=-n}^{n} \mathrm{Tr}\,(A_j + i\Lambda_0/2)^2 + \Re \sum_{j=-n}^{n} \log\det(A_j - i\Lambda_0/2) \Big\} \\ & \times \exp\Big\{ -\frac{W^2}{2} \sum_{j=-n+1}^{n} \mathrm{Tr}\,(V_j A_j V_j^* - A_{j-1})^2 \Big\} \\ & \times \prod_{l=-n}^{n} (a_l - b_l)^2 d\,\mu(U_{-n})\, d\overline{a}\, d\overline{b} \prod_{p=-n+1}^{n} d\mu(V_p). \end{aligned}$$

The integral over the unitary group $U(2)$ can be computed using the well-known Harish Chandra/Itsykson-Zuber formula (see e.g. [13], Appendix 5)

Proposition 1. *Let A be the normal $p\times p$ matrix with distinct eigenvalues $\{a_i\}_{i=1}^p$ and $B = diag\{b_1,\dots,b_p\}$. Then*

$$\int\limits_{U(p)}\int\limits_{\Omega} \exp\Big\{-\frac{1}{2}\mathrm{Tr}\,(A-U^*BU)^2\Big\}\triangle^2(B)f(B)d\,Ud\,B$$
$$=\Big(\prod_{j=1}^p j!\Big)\int\limits_{\Omega}\exp\Big\{-\frac{1}{2}\mathrm{Tr}\,(a_j-b_j)^2\Big\}\frac{\triangle(B)}{\triangle(A)}f(b_1,\dots,b_p)d\,B, \quad (3.10)$$

where $f(B)$ is any symmetric function of $\{b_j\}_{j=1}^p$ in the symmetric domain Ω, $d\,B = \prod\limits_{j=1}^p d\,b_j$ and $\triangle(A)$, $\triangle(B)$ are the Vandermonde determinants for the eigenvalues $\{a_i\}_{i=1}^p$, $\{b_i\}_{i=1}^p$ of A and B.

We get (recall that $A_j = \mathrm{diag}\,\{a_j,b_j\}$, $j=-n,\dots,n$)

$$\begin{aligned}|\Sigma_c| \le& 2^{2n}W^{-4n}e^{2(2n+1)c_0}\int\limits_{\Omega_\delta^C}\exp\Big\{-\frac{W^2}{2}\sum_{j=-n+1}^n(a_j-a_{j-1})^2-\frac{W^2}{2}\sum_{j=-n+1}^n(b_j-b_{j-1})^2\Big\}\\
&\times\exp\Big\{-\sum_{j=-n}^n(f_*(a_j)+f_*(b_j))\Big\}\,|(a_{-n}-b_{-n})(a_n-b_n)|\,d\overline{a}\,d\overline{b}\\
=&2^{2n}W^{-8n-4}e^{2(2n+1)c_0}\int\limits_{W\Omega_\delta^C}\exp\Big\{-\frac{1}{2}\sum_{j=-n+1}^n(a_j-a_{j-1})^2-\frac{1}{2}\sum_{j=-n+1}^n(b_j-b_{j-1})^2\Big\}\\
&\times\exp\Big\{-\sum_{j=-n}^n(f_*(a_j/W)+f_*(b_j/W))\Big\}\,|(a_{-n}-b_{-n})(a_n-b_n)|\,d\overline{a}\,d\overline{b},\end{aligned}\quad (3.11)$$

where

$$f_*(x) := \Re\big((x+i\lambda_0/2)^2/2-\log(x-i\lambda_0/2)\big)-c_0, \quad c_0 := \frac{1}{2}-\frac{\lambda_0^2}{4}. \quad (3.12)$$

We need

Lemma 1. *The function $f_*(x)$ for $x\in\mathbb{R}$ attains its minimum at $x=a_\pm$, where $a_\pm$ is defined in (3.2). Moreover, $f_*(a_\pm)=0$ and if $x\notin U_\delta(a_\pm) := (a_\pm-\delta, a_\pm+\delta)$ for sufficiently small δ, then*

$$f_*(x) \ge C\delta^2. \quad (3.13)$$

In addition, we have for $x\in(-\infty,\delta)$

$$f_*(x) \ge \alpha\,(x-a_-)^2, \quad (3.14)$$

where α is some positive constant. Similar inequality holds for $x\in(-\delta,+\infty)$ (with a_+ instead of a_-).

Consider $a_{-n},\dots,a_n$. First note that the integral over the configurations of $\{a_j\}_{j=-n}^n$ such that $|a_j-a_{j-1}|\ge Wn^{-\varepsilon}$ with sufficiently small ε satisfies bound (3.3). Indeed,

$$\frac{1}{2}\sum_{j=-n+1}^{n}(a_j-a_{j-1})^2\ge CW^2n^{-2\varepsilon}=Cn^{1+\theta-2\varepsilon}.$$

Besides, Lemma 1 yields

$$\begin{aligned} f_*(x)&\ge\alpha\,(x-a_-)^2, \qquad & x\le a_-,\\ f_*(x)&\ge\alpha\,(x-a_+)^2, \qquad & x\le a_+, \end{aligned}$$

and hence the integral in (3.11) over $\prod_{q=-n}^n da_q$ can be bounded by

$$C_1W^{2n+1}\exp\{-Cn^{1+\theta-2\varepsilon}\}\le\exp\{-Cn^{1+\theta-2\varepsilon}/2\}$$

for any $\varepsilon<\theta/2$.

Thus we need to study the configurations such that $|a_j-a_{j-1}|\le Wn^{-\varepsilon}$, $j=-n+1,\dots,n$. Without loss of generality let $a_{-n}<0$. Let l_1 be the first number such that $a_{l_1}>\delta W$. Consider the nearest to l_1 indexes $p_1<l_1$ and $q_1>l_1$ such that $a_{p_1}\le 0$, $a_{q_1}\le 0$. The sequence $a_{p_1+1},\dots,a_{q_1-1}$ we will call “the peak”. Remove from the sum $\sum\limits_{j=-n+1}^{n}(a_j-a_{j-1})^2$ the terms $(a_{p_1+1}-a_{p_1})^2$ and $(a_{q_1}-a_{q_1-1})^2$ (the integral becomes larger). Then take the first number $l_2>q_1$ such that $a_{l_2}>\delta W$ and the nearest to l_2 indexes $p_2<l_2$ and $q_2>l_2$ such that $a_{p_2}\le 0$, $a_{q_2}\le 0$ and again remove the terms $(a_{p_2+1}-a_{p_2})^2$ and $(a_{q_2}-a_{q_2-1})^2$, and so on (the last “peak” can be from a_{p_j+1} to a_{-n}). Assume that we obtain k of such “peaks”.

Consider one of them. Let it has $m+1$ positive numbers $a_{p_r+1},\dots,a_{p_r+m+1}=a_{q_r-1}$. Since $|a_j-a_{j-1}|\le Wn^{-\varepsilon}$, we have $m\ge n^{\varepsilon}\delta$ and taking into account that $a_{p_r}<0$, we have $a_{p_r+1}/W-a_+>\delta$, $|a_{l_r}-a_{p_r+1}|\ge\delta W/2$. Set

$$\mu_\gamma^{(m)}(x)=\exp\Big\{-\frac{1}{2}\sum_{j=2}^{m}(x_j-x_{j-1})^2-\frac{\gamma}{W^2}\sum_{j=1}^{m}x_j^2\Big\}. \tag{3.15}$$

Lemma 2. *For any $\gamma\in\mathbb{C}$, $\Re\gamma>0$ we have*

(1)

$$Z_\gamma^{(m)}:=\int\mu_\gamma^{(m)}(x)\prod_{q=1}^{m}dx_q=(2\pi)^{m/2}\Big(\frac{\sqrt{2\gamma}}{W}\sinh\frac{m\sqrt{2\gamma}}{W}\Big)^{-1/2}(1+o(1)) \tag{3.16}$$

Moreover,if we set

$$G^{(m)}(\gamma)=\left(-\Delta+\frac{2\gamma}{W^2}\right)^{-1}, \tag{3.17}$$

then

$$G_{ii}^{(m)}(\gamma)\le\frac{C_\gamma W}{\sqrt{2\gamma}}\tanh^{-1}\frac{m\sqrt{2\gamma}}{W}(1+o(1)) \tag{3.18}$$

(2) $$\frac{|Z_\gamma^{(m)} - Z_{\delta,\gamma}^{(m)}|}{|Z_\gamma^{(m)}|} := |Z_\gamma^{(m)}|^{-1}\Big|\int\limits_{\max|x_i|>\delta W} \mu_\gamma^{(m)}(x)\prod_{q=1}^m dx_q\Big| \le C_1\, e^{-C_2\delta^2 W}, \quad W\to\infty$$
where $m > CW$, $\delta = W^{-\kappa}$ for sufficiently small κ.
In addition, for any m
$$|Z_\gamma^{(m)}|^{-1}\Big|\int\limits_{|x_k-x_1|>\delta W} \mu_\gamma^{(m)}(x)\prod_{q=1}^m dx_q\Big| \le C_1\, e^{-C_2\delta^2 W}, \quad W\to\infty.$$

(3) *Let $m > C_1 W$, $k \le Cm/W$, $S = \{i_1,\dots,i_s\}\subset\{1,\dots,m\}$, and $\sum\limits_{l=1}^{s} k_{i_l} = 3k$, where $k_l \in \{3,\dots,k\}$. Then*
$$|Z_\gamma^{(m)}|^{-1}\Big|\int\limits_{\max|x_i|>\delta W}\prod_{j\in S}(x_j/W)^{k_j}\cdot\mu_\gamma^{(m)}(x)\prod_{q=1}^m dx_q\Big| \le e^{-C_1\delta^2 W}, \quad m, W\to\infty,$$
where $\delta = W^{-\kappa}$ for sufficiently small κ.

Since $a_{p_r+1},\dots,a_{p_r+m+1} > 0$, according to Lemma 1 we can write
$$f_*(a_{p_r+s}/W) \ge \alpha\,(a_{p_r+s}/W - a_+)^2, \quad s = 1,\dots,m+1.$$
Using this inequality in the r.h.s. of (3.11), we can apply Lemma 2 to the integral over $da_{p_r+1},\dots,da_{p_r+m+1}$. This gets the bound $(2\pi)^{m/2}W\exp\{-m\sqrt{2\alpha}/(2W) - C\delta^2 W\}$ (recall that $a_{p_r+1}/W - a_+ > \delta$ and $|a_{l_r} - a_{p_r+1}| \ge \delta W/2$). Hence, for integrals over variables which make k "peaks" we obtain the bound
$$(2\pi)^{\sum m_i/2}\, W^k \exp\{-\sqrt{2\alpha}\sum m_i/(2W)\}\exp\{-C\delta^2 Wk\}.$$
By the same way we can estimate the integral over $a_{q_l},\dots,a_{p_{l+1}}$ (i.e. over a_j's that lying between two "peaks") by $(2\pi)^{s/2}W\exp\{-s\sqrt{2\alpha}/(2W)\}$, where $s = p_{l+1} - q_l + 1$. Finally, the whole integral with k "peaks" can be bounded by
$$(2\pi)^n W^{2k+1}\exp\{-\sqrt{2\alpha}\,n/(2W)\}\exp\{-\delta^2 Wk\}.$$
The number of such configurations is smaller than $\binom{2n+1}{2k}$ (since the number of choices of the "beginnings" and "ends" of k "peaks" is $\binom{2n+1}{2k}$). Hence, we get the bound for the integral over configurations with $k > 0$ in (3.11):
$$Z_\gamma^{-1}\cdot(2\pi)^n\exp\{-\sqrt{2\alpha}\,n/(2W)\}\sum_{k=1}^n\binom{2n+1}{2k}W^{2k+1}\exp\{-\delta^2 Wk\}$$
$$\le \sqrt{W}e^{-C\delta^2 W + Cn/W}((1+We^{-\delta^2 W})^n - 1) \le e^{-C_1\delta^2 W}.$$
Therefore, the main contribution to (3.11)is given by the configurations without "peaks". For such configurations
$$f_*(a_j) \ge \alpha\,(a_j - a_+)^2, \quad j = -n,\dots,n,$$
and thus Lemma 2 proves (3.3).

3.3 Calculation of Σ

3.3.1 $\Sigma_\pm$ and $\Sigma_\mp$

Consider the δ-neighborhood of the point $(\overline{a}_+, \overline{a}_-)$ with $\overline{a}_\pm$ of (3.2) and $\delta = W^{-\kappa}$

We can write for $x \in U_\delta(a_\pm)$

$$f(x) = \frac{1}{2}(x + i\lambda_0/2)^2 - \log(x - i\lambda_0/2) - r_\pm$$
$$=: c_\pm(x - a_\pm)^2 + \varphi_\pm(x - a_\pm). \tag{3.19}$$

where

$$c_\pm = 1 - \frac{\lambda_0^2}{4} \pm \frac{i\lambda_0}{2} \cdot \sqrt{1 - \frac{\lambda_0^2}{4}}, \quad r_\pm = \frac{1}{2}(a_\pm + i\lambda_0/2)^2 - \log(a_\pm - i\lambda_0/2). \tag{3.20}$$

It is easy to check that $\Re r_\pm = c_0$, where c_0 is defined in (3.12).

Doing the change $a_j - a_+ = \tilde{a}_j/W$, $b_j - a_- = \tilde{b}_j/W$ in (2.14) we obtain (recall that $a_\pm = \pm\pi\rho(\lambda_0)$)

$$\begin{aligned}\Sigma_\pm =& W^{-2(2n+1)}2^{2n}e^{2(2n+1)c_0 + i\pi(\xi_1-\xi_2)} \int\limits_{|\tilde{a}_j|,|\tilde{b}_j| \le W^{1-\kappa}} \int\limits_{U(2)} \mu_{c_+}(a)\mu_{c_-}(b) \\ &\times e^{W^2 \sum\limits_{j=-n+1}^{n} \mathrm{Tr}\left(V_j^*(L+\tilde{A}_j/W)V_j(L+\tilde{A}_{j-1}/W) - (L+\tilde{A}_j/W)(L+\tilde{A}_{j-1}/W)\right)} \\ &\times e^{-\sum\limits_{k=-n}^{n}(\varphi_+(\tilde{a}_k/W)+\varphi_-(\tilde{b}_k/W)) + \frac{i}{n\rho(\lambda_0)}\sum\limits_{p=-n}^{n}\left(\mathrm{Tr}\,(U_{-n}P_p)^*(L+\tilde{A}_p/W)\,(U_{-n}P_p)\hat{\xi} - \mathrm{Tr}\,L\hat{\xi}\right)} \\ &\times \prod_{l=-n}^{n}(a_+ - a_- + (\tilde{a}_l - \tilde{b}_l)/W)^2 d\mu(U_{-n}) \prod_{q=-n+1}^{n} d\mu(V_q)\, d\overline{a}\, d\overline{b}(1 + o(1)),\end{aligned} \tag{3.21}$$

where $L = \mathrm{diag}\,\{a_+, a_-\}$, $\tilde{A}_j = \mathrm{diag}\,\{\tilde{a}_j, \tilde{b}_j\}$, and $\mu_\gamma(a)$ is defined in (3.6).

We will use bellow the following form of Wick's theorem: for any smooth function f

$$\langle x_{i_1} f(x_{i_1}, \ldots, x_{i_p})\rangle = \sum_{j=1}^{p}\langle x_{i_1}x_{i_j}\rangle\langle \partial f(x_{i_1}, \ldots, x_{i_p})/\partial x_{i_j}\rangle. \tag{3.22}$$

The same is valid for $\langle \ldots \rangle_*$, where $\langle \ldots \rangle$, $\langle \ldots \rangle_*$ are defined in (3.7). Set

$$M = -\triangle + \gamma/W^2 = (2+\gamma/W^2)I - \tilde{M}, \quad M_* = -\triangle + \Re\gamma/W^2 = (2+\Re\gamma/W^2)I - \tilde{M}, \tag{3.23}$$

where

$$\tilde{M} = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$\langle x_i x_j\rangle = (M^{-1})_{ij}, \quad \langle x_i x_j\rangle_* = (M_*^{-1})_{ij}.$$

Besides, since all entries of $\tilde M$ are positive and $\Re\gamma > 0$,

$$\big|(M^{-1})_{ij}\big| = \Big|\sum_{k=0}^{\infty}\frac{(\tilde M^k)_{ij}}{(2+\gamma/W^2)^{k+1}}\Big|$$
$$\le \sum_{k=0}^{\infty}\frac{(\tilde M^k)_{ij}}{|2+\gamma/W^2|^{k+1}} \le \sum_{k=0}^{\infty}\frac{(\tilde M^k)_{ij}}{(2+\Re\gamma/W^2)^{k+1}} = (M_*^{-1})_{ij}.$$

This and (3.22) yield (3.9).

First step: calculation of $\Big\langle \exp\Big\{-\sum\limits_{j=-n}^{n}\varphi_\pm(x_j/W)\Big\}\Big\rangle_0$

Define

$$E_n[g] := \exp\Big\{-\sum_{j=-n}^{n} g(x_j/W)\Big\} \tag{3.24}$$

for any function $g:\mathbb{R}\to\mathbb{C}$.

We are going to prove that

$$\big|\big\langle E_n[\varphi_\pm]\big\rangle_0 - 1\big| = o(1), \tag{3.25}$$

where $\varphi_\pm$ are defined in (3.19).

The key point of the first step is the lemma:

Lemma 3. *Let* $g = \sum\limits_{j=3}^{d} c_j x^j$, $c_j \in \mathbb{R}$. *Then we have*

$$\big|\big\langle E_n[g]\big\rangle_{0,*} - 1\big| = o(1), \quad n\to\infty. \tag{3.26}$$

Proof. Since $e^x - 1 \ge x$, we have

$$\big\langle E_n[g]\big\rangle_{*,0} - 1 \ge \big\langle \sum_{j=-n}^{n} g(x_j/W)\big\rangle_{*,0} = \big\langle \sum_{j=-n}^{n} g(x_j/W)\big\rangle_{*} + o(1),$$

where in the last equality we use the third assertion of Lemma 2. Using the Wick's theorem (3.22) and $(M_*^{-1})_{ii} = CW$ (see the assertion (1) of Lemma 2), we can write

$$\big\langle (x_j/W)^{2l}\big\rangle_* = O(W^{-l}),$$

and hence

$$\big\langle \sum_{j=-n}^{n} g(x_j/W)\big\rangle_* = O((2n+1)/W^2) = o(1).$$

Let us prove that

$$\big\langle E_n[g]\big\rangle_{*,0} - 1 \le \varepsilon_{1,n}\big\langle E_n[g]\big\rangle_{*,0} \Rightarrow \big\langle E_n[g]\big\rangle_{*,0} - 1 \le 2\varepsilon_{1,n}, \tag{3.27}$$

where $\varepsilon_{1,n} = o(1)$, as $n\to\infty$.

Note that if we choose $s_\kappa > 3$ such that

$$W^{-\kappa s_\kappa} \le W^{-2}. \tag{3.28}$$

then for any $p > s_\kappa/2$ and for $x_j \in (-\delta W, \delta W)$

$$\sum_j (x_j/W)^{2p} < n/W^2 = o(1),$$

and thus if we replace $g(x)$ by $g(x) - Cx^{2p}$ with any C, then $E_n[g]$ will be changed by $E_n[g](1+o(1))$. Since it is easy to see that we can choose C such that $c_0x^2/2+g(x)-Cx^{2p}$ has only one minimum $x = 0$ in $\mathbb{R}$, without loss of the generality we can assume that $c_0x^2/2 + g(x) \ge c_0x^2/4$. Moreover, $c_0x^2/2 + g(x) \le c_0x^2$ for $x \in (-\delta, \delta)$. This and assertions (1), (2) of Lemma 2 give

$$\frac{\displaystyle\int_{\max|x_i|>\delta W} E_n[g]\mu_{c_0}(x)dx}{\displaystyle\int_{\max|x_i|\le\delta W} E_n[g]\mu_{c_0}(x)dx} \le \frac{\displaystyle\int_{\max|x_i|>\delta W} \mu_{c_0/2}(x)dx}{\displaystyle\int_{\max|x_i|\le\delta W} \mu_{2c_0}(x)dx} \le e^{Cn/W - W\delta^2} = o(1),$$

because $\delta = W^{-\kappa}$ with $\kappa < \theta/2$. Thus,

$$\big\langle E_n[g]\big\rangle_{*,0} = \big\langle E_n[g]\big\rangle_* + o(1). \tag{3.29}$$

Since for $x \in \mathbb{R}$

$$e^x \le 1 + xe^x,$$

we can write using the Wick's theorem (3.22)

$$\begin{aligned}
\big\langle E_n[g]\big\rangle_* - 1 &\le \sum_{i_1} \big\langle g(x_{i_1}/W)\cdot E_n[g]\big\rangle_* = \sum_{i_1}\sum_{l=3}^{d} \Big\langle \frac{c_l x_{i_1}^l}{W^l}\cdot E_n[g]\Big\rangle_* \\
&\le \sum_{i_1}\sum_{l=3}^{d} \frac{(l-1)|c_l|\langle x_{i_1}^2\rangle_*}{W^2}\Big|\Big\langle \frac{x_{i_1}^{l-2}}{W^{l-2}}\cdot E_n[g]\Big\rangle_*\Big| \\
&+ \sum_{i_1,i_2}\sum_{l=3}^{d} \frac{|c_l|\langle x_{i_1}x_{i_2}\rangle_*}{W^2}\Big|\Big\langle \frac{x_{i_1}^{l-1}}{W^{l-1}}\cdot g'\Big(\frac{x_{i_2}}{W}\Big)\cdot E_n[g]\Big\rangle_*\Big| \\
&\le \sum_{i_1}\sum_{l=4}^{d} \frac{(l-1)(l-3)|c_l|\langle x_{i_1}^2\rangle_*^2}{W^4}\Big|\Big\langle \frac{x_{i_1}^{l-4}}{W^{l-4}}\cdot E_n[g]\Big\rangle_*\Big| \\
&+ \sum_{i_1,i_2}\sum_{l=3}^{d} \frac{(2l-3)|c_l|\langle x_{i_1}x_{i_2}\rangle_*\langle x_{i_1}^2\rangle_*}{W^4}\Big|\Big\langle \frac{x_{i_1}^{l-3}}{W^{l-3}}\cdot g'\Big(\frac{x_{i_2}}{W}\Big)\cdot E_n[g]\Big\rangle_*\Big| \\
&+ \sum_{i_1,i_2}\sum_{l=3}^{d} \frac{|c_l|\langle x_{i_1}x_{i_2}\rangle_*^2}{W^4}\Big|\Big\langle \frac{x_{i_1}^{l-2}}{W^{l-2}}\cdot g''\Big(\frac{x_{i_2}}{W}\Big)\cdot E_n[g]\Big\rangle_*\Big| \\
&+ \sum_{i_1,i_2,i_3}\sum_{l=3}^{d} \frac{|c_l|\langle x_{i_1}x_{i_2}\rangle_*\langle x_{i_1}x_{i_3}\rangle_*}{W^4}\Big|\Big\langle \frac{x_{i_1}^{l-2}}{W^{l-2}}\cdot g'\Big(\frac{x_{i_2}}{W}\Big) g'\Big(\frac{x_{i_3}}{W}\Big)\cdot E_n[g]\Big\rangle_*\Big| = \dots
\end{aligned}$$

Every time having $\langle x_{i_1}^{m_1} \dots x_{i_k}^{m_k} E_n[g] \rangle_*$ we take x_{i_l} with the smallest index l and find the pair to it according to (3.22). In such a way we make $2ds_\kappa$ steps, where s_κ is defined in (3.28) and d is a degree of the polynomial g. All terms have the form

$$\sum_{i_1,\dots,i_{p+l}} G(x_{i_1},\dots,x_{i_{p+l}}) \left| \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \dots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_* \right|,$$

where $\alpha_{p+1},\dots,\alpha_{p+l} \in \mathbb{N}$ are bounded by some absolute constant (since we make the finite number of steps), $\alpha = \alpha_{p+1} + \dots + \alpha_{p+l}$, and $G(x_{i_1},\dots,i_{p+l})$ is the product of the expectations of some partition by pairs of $x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_{p+l}}^{k_{p+l}}$ with $k_j \geq 3$, $j = 1,\dots,p$ and $k_j \geq 1$, $j = p+1,\dots,p+l$ (all k_j's are bounded by some absolute constant), with some bounded positive coefficient and divided by $W^{k_1+\dots+k_{p+l}}$. These partitions we can imagine like graphs with vertex $i_1,\dots,i_{p+l}$, where $p+l \leq 2ds_\kappa$ and the degree of each vertex i_j is grater than 3 for $j \leq p$ and is grater than 1 for $j = p+1,\dots,p+l$ (these graphs can contain loops, i.e. edges like (i_j,i_j), and multiple edges). Let H be one of such graphs. Any $\langle x_i x_j \rangle_*$ gives $(M_*^{-1})_{ij}$. Thus, any loop gives factor $(M_*^{-1})_{ii} = CW(1+o(1))$ (see the assertion (1) of Lemma 2). Moreover, according the Cauchy-Schwarz inequality we have

$$(M_*^{-1})_{ij} \leq (M_*^{-1})_{ii}^{1/2} (M_*^{-1})_{jj}^{1/2}.$$

Thus, we can remove the edge (j_1,j_2) from any cycle $(j_1,j_2,\dots,j_r,j_1)$ $(r \neq 1)$ and change it into two "semiloops" (j_1,j_1), (j_2,j_2) (i.e. we must remember that these "semiloops" give $|(M_*^{-1})_{ii}|^{1/2}$ instead of $(M_*^{-1})_{ii}$ and one "semiloop" gives the contribution one to the degree of the vertex, two "semiloops" in the same vertex give the loop). In this way we transform graph H to the collection of the trees H_0 with some loops and "semiloops" (the degree of each vertex still the same as in H). How does the graph H_0 look like? According to our procedure every time we take x_{i_l} with the smallest index l. Therefore, we take the vertex i_1, then connect it with itself or with the vertex i_2, then, if we still have x_{i_1} in $\langle x_{j_1}^{m_1} \dots x_{j_k}^{m_k} E_n[g] \rangle_*$ we take it again and so on. At some moment we can get the additional $\langle E_n[g] \rangle_*$. This means that in this step we obtain the connected component with the vertexes $i_1,\dots,i_k$ of the degree grater then 3. Then, using the inequality $e^x \leq 1 + xe^x$, we continue the procedure with $\langle E_n[g] \rangle_*$. At some step we can again obtain the connection to one of the vertexes $i_1,\dots,i_k$, but then we must take this vertex and looking for the pair to it. Hence, it is easy to see that we can obtain several connected component, but all vertexes $i_{p+1},\dots,i_{p+l}$ lie in the same (we will call it the last) connected component of H thus of H_0.

Since we make the finite number of steps, there are only a finite number of graphs H such that corresponding graphs H_0 are equal, hence we can consider the sum over H_0 instead of H. Let $G_0(x_{i_1},\dots,i_{p+l})$ be the function, which corresponds to the new graph H_0.

Note that according to (3.29)

$$\begin{aligned} &\left| \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \dots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_* \right| \leq \left| \left\langle \frac{x_{i_{p+1}}^{\alpha_{p+1}} x_{i_{p+2}}^{\alpha_{p+2}} \dots x_{i_{p+l}}^{\alpha_{p+l}}}{W^\alpha} E_n[g] \right\rangle_{0,*} + o(1) \right| \\ &\leq \frac{1}{W^{\varepsilon\alpha}} \left\langle E_n[g] \right\rangle_{0,*} + o(1) = \frac{1}{W^{\varepsilon\alpha}} \left\langle E_n[g] \right\rangle_* + o(1). \end{aligned}$$

Therefore, we are interested in

$$\sum_{i_1,\dots,i_{p+l}} W^{-\varepsilon\alpha} G_0(x_{i_1},\dots,x_{i_{p+l}}). \tag{3.30}$$

Since $(1,\dots,1)$ is an eigenvector for M_* of (3.23) with eigenvalue $W^2/\Re\gamma$, we have

$$\sum_j (M_*^{-1})_{ij} = \frac{W^2}{\Re\gamma}, \quad i=-n,\dots,n. \tag{3.31}$$

Consider any (but not the last) connected component of H_0. Let it has k vertexes $i_1,\dots,i_k$. This component is a tree with some loops and "semiloops". The degree m_{i_j} of each vertex i_j is not smaller than 3, hence $m := \sum m_{i_j} \ge 3k$ and the equality holds iff all degrees of the vertexes are 3. Let us consider the sum over $i_1,\dots,i_k$. Any loop or "semiloop" gives W or $W^{1/2}$ respectively. Thus, since the tree has $k-1$ edges, all loops give the contribution $W^{m/2-k+1}$. Using (3.31), we obtain that the contribution of the tree's edges is $nW^{2(k-1)}$. Therefore, since any x_{i_j} has also the coefficient W^{-1}, we obtain that the sum over $i_1,\dots,i_k$ gives $n/W^{m/2-k+1}$. Evidently m is even, and hence $m/2-k+1$ is integer. Moreover, since $m\ge 3k$ we get $m/2-k+1 \ge m/6+1$, and thus

$$\begin{aligned} &m/2-k+1 \ge 2, \quad m\le 6,\\ &m/2-k+1 \ge 3, \quad 6<m\le 12,\\ &m/2-k+1 \ge 4, \quad m>12. \end{aligned} \tag{3.32}$$

By the same way we get that the contribution of the last connected component is not more than $n/W^{m/2-k+1}$, and $m\ge 2(k-1)$, i.e. $m/2-k+1\ge 0$. In addition we have for the last component

$$m+\alpha \ge 3k \Rightarrow k \le (m+\alpha)/2. \tag{3.33}$$

Assume that H_0 has q connected components with the numbers of vertexes $a_1,\dots,a_q$ and with the sum of the vertex degrees $b_1,\dots,b_q$ respectively. Then (3.30) is bounded by

$$\frac{1}{W^{\varepsilon\alpha}} \cdot \frac{n}{W^{b_q/2-a_q+1}} \cdot \prod_{j=1}^{q-1} \frac{n}{W^{b_j/2-a_j+1}}.$$

According to (3.32), if two of $b_1,\dots,b_{q-1}$ are greater than 6 or one of $b_1,\dots,b_{q-1}$ is greater than 12, then the product is bounded by $W^{-\varepsilon(q-1)}\cdot W^{-2}$, and, since $b_q/2-a_q+1\ge 0$, we have that the sum (3.30) is bounded by $W^{-\varepsilon(q+\alpha)}$. If $b_q/2-a_q+1\ge 2$, then we also have the bound $W^{-\varepsilon(q+\alpha)}$. Thus, it remains to consider the case $b_q/2-a_q+1=0$ or 1 and one of $b_1,\dots,b_{q-1}$ is not grater then 12 and others are not grater then 6. Besides, $b_1+\ldots+b_q = 4ds_\kappa$, since we did $2ds_\kappa$ steps and thus obtained $2ds_\kappa$ edges. Hence, since $d\ge 3$ and according to (3.33) $b_q/2-a_q+1\le 1$ yields $b_q\le 2\alpha$, we have

$$12s_\kappa \le 4ds_\kappa = b_1+\ldots+b_q \le 6(q-2)+12+2\alpha = 6q+2\alpha.$$

Therefore, $q>s_\kappa$ or $\alpha>s_\kappa$, where s_κ is defined in (3.28). In both cases we get the bound $n\cdot W^{-\varepsilon(q-1+\alpha)} \le n\cdot W^{-s\varepsilon} = o(1)$, which gives (3.27) with $\langle\ldots\rangle_*$ and thus with $\langle\ldots\rangle_{*,0}$ (see (3.29)).

□

Come back to the proof of (3.25). We can write for $x \in (-\delta, \delta)$

$$\widetilde{\varphi}_{\pm}(x) := \exp\{-\varphi_{\pm}(x)\} - 1 = \sum_{l=3}^{\infty} \phi_l x^l,$$

where $|\phi_l| \le (C_0)^l$. Thus

$$|\langle E_n[\varphi_\pm]\rangle_0 - 1| = \Big|\langle \prod_{j=-n}^{n} (1 + \sum_{l=3}^{\infty} \phi_l x^l)\rangle_0 - 1\Big| = \Big|\sum_{k=3}^{\infty} \Sigma_k^0\Big|, \tag{3.34}$$

where Σ_k^0, Σ_k is the sum of all terms $\langle\prod_{l=1}^s (\phi_{k_l} x_{i_l}^{k_l}/W^{k_l})\rangle_0$ and $\langle\prod_{l=1}^s (\phi_{k_l} x_{i_l}^{k_l}/W^{k_l})\rangle$ respectively with $k_1 + \ldots + k_s = k$, $k_i \in \{3, \ldots, k\}$ ($\Sigma_{k,*}^0$, $\Sigma_{k,*}$ are defined by the same way with $\langle\ldots\rangle_*$ instead of $\langle\ldots\rangle$ and with $|\phi_l|$ instead of ϕ_l).

Denote also

$$S_s^0 = \sum_{i_1<\ldots<i_s} \langle\prod_{l=1}^{s} |x_{i_l}/W|^3\rangle_{0,*}.$$

According to (4.9) (see below) we have

$$|\langle(\phi_{k_1} x_{i_1}^{k_1}/W^{k_1})\ldots(\phi_{k_s} x_{i_s}^{k_s}/W^{k_s})\rangle_0| \le (C_0)^k \delta^{k-3s} e^{Cn/W} \langle\prod_{l=1}^{s} |x_{i_l}/W|^3\rangle_{0,*}.$$

Hence, since the number of partitions of k to s non-zero summands is not grater than $\binom{k}{s}$, we obtain

$$|\Sigma_k^0| \le e^{Cn/W} (C_0)^k \sum_{s=1}^{k/3} \binom{k}{s} \delta^{k-3s} S_s^0 \le e^{Cn/W} (2C_0)^k \sum_{s=1}^{k/3} \delta^{k-3s} S_s^0. \tag{3.35}$$

Note now that

$$|x|^3 \le \frac{p^{-1}x^2 + px^4}{2}, \tag{3.36}$$

and hence again according to (4.9) (see below) we get for any $p > 0$

$$S_s^0 \le \sum_{i_1<\ldots<i_s} \Big\langle \prod_{l=1}^{s} \frac{p^{-1}x_{i_l}^2/W^2 + px_{i_l}^4/W^4}{2}\Big\rangle_{0,*} := \widetilde{S}_s^0. \tag{3.37}$$

Besides,

$$1 + q \cdot \frac{p^{-1}x^2 + px^4}{2} \le (1 + \frac{qx^2}{2p})(1 + \frac{pqx^4}{2}) \le e^{qx^2/2p}(1 + \frac{pqx^4}{2}),$$

and thus, taking in account (4.9) (see below), for any $p, q > 0$ such that $q/p < c_0$ with c_0 of (3.12) we have

$$\begin{aligned}
1 + \sum_{k=1}^{2n+1} q^k \widetilde{S}_k^0 &= \Big\langle \prod_{j=-n}^{n} \Big(1 + q \cdot \frac{p^{-1}x_j^2/W^2 + px_j^4/W^4}{2}\Big)\Big\rangle_{0,*} \\
&\le \Big\langle e^{q/2p \cdot \sum_j x_j^2/W^2} \prod_{j=-n}^{n} \Big(1 + \frac{pqx_j^4/W^4}{2}\Big)\Big\rangle_{0,*} \\
&\le e^{C_{p,q}n/W} \Big\langle \prod_{j=-n}^{n} \Big(1 + \frac{pqx_j^4/W^4}{2}\Big)\Big\rangle_{0,c_0-q/p} \le Ce^{C_{p,q}n/W},
\end{aligned}$$

where the last inequality holds in view of Lemma 3 ($\langle \dots \rangle_{0,c_0-q/p}$ means (3.7) with $\gamma = c_0 - q/p$). This gives

$$\widetilde{S}_k^0 \le e^{C_{p,q}n/W}/q^k,$$

and we have from (3.37) for $k > Cn/W$ with sufficiently big C

$$S_k^0 \le e^{(C_{p,q}+C_1)n/W} q^{-k}. \tag{3.38}$$

Take $q > (2|C_0|e)^3$. Then (3.39) and (3.38) yield for $k > C_1 n/W$

$$|\Sigma_k^0| \le e^{Cn/W} \sum_{s=1}^{k/3} (2C_0)^k \delta^{k-3s} q^{-s} \le \frac{e^{Cn/W}(2C_0)^k}{q^{k/3}} \sum_{s=1}^{k/3} (\delta^3 q)^{k/3-s} \le 2e^{Cn/W-k}. \tag{3.39}$$

This and (3.34) imply

$$|\langle E_n[\varphi_\pm]\rangle_0 - 1| \le \Big| \sum_{k=3}^{Cn/W} \Sigma_0^k \Big| + e^{-C_1 n/W}. \tag{3.40}$$

Taking into account that the number of distributions of k items into n boxes is $\binom{n+k-1}{k}$ and using the assertion (3) of Lemma 2 we get

$$\begin{aligned} &|Z_\gamma|^{-1} \Big| \int\limits_{\max|x_i|>\delta W} \sum_{s=1}^{k/3} \sum_{k_1,\dots,k_s} \sum_{i_1<\dots<i_s} \frac{|x_{i_1}^{k_1} \dots x_{i_s}^{k_s}|}{W^k} \mu_\gamma(x)dx \Big| \\ &\le e^{-C\delta^2 W} \binom{n+k-1}{k} \le e^{2k\log(n/k)-C\delta^2 W} \le e^{-C\delta^2 W/4}, \end{aligned}$$

where the second sum in the first line is over all collections $\{k_i\}_{i=1}^s$, $\sum k_i = k$, $k_i \in \{3, \dots, k\}$. This yields

$$\Sigma_k = \Sigma_k^0 + e^{-C\delta^2 W/4}, \quad \Sigma_{k,*} = \Sigma_{k,*}^0 + e^{-C\delta^2 W/4}, \quad k \le Cn/W$$

and thus by (3.9) we have

$$\begin{aligned} &| \sum_{k=1}^{Cn/W} \Sigma_k^0 | = | \sum_{k=1}^{Cn/W} \Sigma_k | + e^{-C\delta^2 W/4} \le \sum_{k=1}^{Cn/W} \Sigma_{k,*} + e^{-C\delta^2 W/4} \\ &\le \sum_{k=1}^{Cn/W} \Sigma_{k,*}^0 + 2e^{-C\delta^2 W/4} \le \langle \prod_{i=-n}^{n} (1 + \sum_{l=3}^{\infty} |\phi_l| x_i^l / W^l) - 1 \rangle_{*,0} + 2e^{-C\delta^2 W/4}. \end{aligned} \tag{3.41}$$

Since $|\phi_l| \le (C_0)^l$, there exists C such that

$$1 + \sum_{l=3}^{\infty} |\phi_l| x^l / W^l \le e^{C(x^3/W^3 + x^4/W^4)}, \quad x \in (-\delta W, \delta W). \tag{3.42}$$

This, Lemma 3 and (3.40) yield (3.25).

Remark 1. *1) Using*

$$|\phi_2|\, x^2 \le \frac{p^{-1}x^2 + px^4}{2}, \quad |\phi_2| = o(1),$$
$$|x|/W \le \frac{n^{-1} + (n/W^2)\,|x|^2}{2} \le \frac{p^{-1}x^2 + px^4}{2} + Cn^{-1}$$

instead of (3.36), we can prove (3.41) for the series started from $l=1$ *with* $|\phi_1| \le CW^{-1}$, $|\phi_2| = o(1)$. *Besides, in view of (3.42)*

$$1 + \sum_{l=1}^{\infty} |\phi_l| x^l/W^l \le e^{|\phi_1|\,x/W + (|\phi_2| - |\phi_1|^2/2)\,x^2/W^2 + C(x^3/W^3 + x^4/W^4)}, \quad x \in (-\delta W, \delta W),$$

and hence the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\langle \prod_{i=-n}^{n} (1 + \sum_{l=1}^{\infty} |\phi_l| x_i^l/W^l) \rangle_{*,0} \le \Big\langle \exp\Big\{ \sum_{i=-n}^{n} 2C(x_i^3/W^3 + x_i^4/W^4) \Big\} \Big\rangle_{*,0}^{1/2} \\
&\times \Big\langle \exp\Big\{ \sum_{i=-n}^{n} (2|\phi_i|\, x/W + (2|\phi_2| - |\phi_1|^2)\, x_i^2/W^2 \Big\} \Big\rangle_{*,0}^{1/2} \\
&\le \exp\{C_1 n|\phi_1|^2 + C_2|\phi_2| n/W\}(1 + o(1)) \le \exp\{C|\phi_2| n/W\},
\end{aligned}$$

where to obtain the third line we use Lemma 3 for the first factor and take the Gaussian integral for the second factor.

2) Define the following partial ordering. Let $\Phi_1(x_1, \dots, x_n)$, $\Phi_2(x_1, \dots, x_n)$ *be two analytic functions in some ball with center at* 0, *and let the coefficients of the Taylor expansion of* Φ_2 *are non-negative. Then we write*

$$\Phi_1 \prec \Phi_2 \tag{3.43}$$

if the absolute value of each coefficient of the Taylor expansion of Φ_1 *does not exceed the corresponding coefficient of* Φ_2.

It is easy to see that

$$\Phi_3 \prec \Phi_1, \quad \Phi_4 \prec \Phi_2 \Rightarrow \Phi_3\Phi_4 \prec \Phi_1\Phi_2. \tag{3.44}$$

The discussion above yields that if

$$\Phi_1(s_1, \dots, s_n) - \Phi_1(0, \dots, 0) \prec \prod_{j=1}^{n} (1 + q(s_i)) - 1,$$

where $s_i = s(\tilde{a}_i/W, \tilde{a}_{i+1}/W, \dots, \tilde{a}_{i+k}/W, \tilde{b}_i/W, \tilde{b}_{i+1}/W, \dots, \tilde{b}_{i+k}/W)$ *is a polynomial with* $s(0, \dots, 0) = 0$, k *is an* n*-independent constant, and* $q(s) = \sum_{j=1}^{\infty} |c_j| s^j$ *with* $|c_1| \le W^{-1}$, $|c_2| = o(1)$, $|c_l| \le (C_0)^l$, $l \ge 3$, *then*

$$\big|\langle \Phi_1 \rangle_0\big| \le \langle \prod_{j=1}^{n} (1 + q(s_i^*)) \rangle_{0,*} + e^{-C\delta^2 W},$$

where s_i^* *is obtained from* s_i *by changing the coefficients of* s *to their absolute values.*

Second step: integration over V_j

Denote

$$F(\overline{a},\overline{b},V)=\frac{i}{\rho(\lambda_0)}\sum_{k=-n}^{n}\Big(\mathrm{Tr}\,\big(U_{-n}P_k\big)^*(L+\tilde{A}_p/W)\,(U_{-n}P_k)\hat{\xi}-\mathrm{Tr}\,L\hat{\xi}\Big), \tag{3.45}$$

$$d\,\widetilde{\mu}(V,\widetilde{A})=e^{W^2\sum\limits_{j=-n+1}^{n}\mathrm{Tr}\,\big(V_j^*(L+\tilde{A}_j/W)V_j(L+\tilde{A}_{j-1}/W)-(L+\tilde{A}_j/W)(L+\tilde{A}_{j-1}/W)\big)}\prod_{q=-n+1}^{n}d\mu(V_q),$$

$$I_{\widetilde{\mu}}=\int d\,\widetilde{\mu}(V,\widetilde{A}).$$

According to the Itsykson-Zuber formula

$$I_{\widetilde{\mu}}=W^{-4n}\prod_{q=-n+1}^{n}\frac{1-e^{-W^2(a_+-a_-+(\tilde{a}_q-\tilde{b}_q)/W)(a_+-a_-+(\tilde{a}_{q-1}-\tilde{b}_{q-1})/W)}}{(a_+-a_-+(\tilde{a}_q-\tilde{b}_q)/W)(a_+-a_-+(\tilde{a}_{q-1}-\tilde{b}_{q-1})/W)}. \tag{3.46}$$

We want to integrate the r.h.s. of (3.21) over $d\widetilde{\mu}(V,\widetilde{A})$. To this end we expand the exponent in the fourth line of (3.21) into the series of $|(V_j)_{12}|^2$ (note that $|(V_j)_{12}|^2=|(V_j)_{21}|^2$, $|(V_j)_{11}|^2=|(V_j)_{22}|^2=1-|(V_j)_{12}|^2$). It is easy to see that

$$\begin{aligned}&\int|(V_j)_{12}|^{2s}d\widetilde{\mu}(V,\widetilde{A}) \qquad (3.47)\\ &=W^{-4n}\prod_{q\neq j}\frac{1-e^{-W^2(a_+-a_-+(\tilde{a}_q-\tilde{b}_q)/W)(a_+-a_-+(\tilde{a}_{q-1}-\tilde{b}_{q-1})/W)}}{(a_+-a_-+(\tilde{a}_q-\tilde{b}_q)/W)(a_+-a_-+(\tilde{a}_{q-1}-\tilde{b}_{q-1})/W)}\\ &\quad\times(-1)^s\frac{d^s}{dx^s}\frac{1-e^{-x}}{x}\Big|_{x=W^2(a_+-a_-+(\tilde{a}_{j-1}-\tilde{b}_{j-1})/W)(a_+-a_-+(\tilde{a}_j-\tilde{b}_j)/W)}.\end{aligned}$$

If we differentiate $1-e^{-x}$, then we obtain e^{-x} which is small, since $x\sim W^2(a_+-a_-)^2$ (recall that $|\widetilde{a}_j/W|,|\widetilde{b}_j/W|\le W^{-\kappa}$). Therefore, we should differentiate only $1/x$.

We want to prove that only the terms without $|(V_j)_{12}|^2$ give the contribution. Hence, we want to show that

$$\Big|\Big\langle\big(\exp\{(F(\overline{a},\overline{b},V)-F(0,0,I))/n\}-1\big)\cdot\Pi_1\cdot\Pi_2\Big\rangle_0\Big|=o(1),\quad n\to\infty,$$

where Π_1, Π_2 are the products of the Taylor's series for $\exp\{\varphi_+(\widetilde{a}_j/W)\}$ and for $\exp\{\varphi_-(\widetilde{b}_j/W)\}$. Besides,

$$\exp\{(F(\overline{a},\overline{b},V)-F(0,0,V))/n\}\prec\exp\Big\{\sum_{j=-n}^{n}\frac{C(\widetilde{a}_j+\widetilde{b}_j)}{nW}\Big\}, \tag{3.48}$$

thus we can estimate only the integral of

$$\exp\{(F(0,0,V)-F(0,0,I))/n\}.$$

We can write

$$e^{\frac{1}{n}(F(0,0,V)-F(0,0,I))}=\sum_{p=0}^{\infty}\frac{1}{p!\,n^p}\left(F(0,0,V)-F(0,0,I)\right)^p.$$

Since $\widehat{\xi} = \frac{\xi_1+\xi_2}{2} I_2 + \frac{\xi_1-\xi_2}{2a_+} L$, we have

$$\begin{aligned}
&\mathrm{Tr}\,(U_{-n}P_k)^* L\,(U_{-n}P_k)\hat{\xi} - \mathrm{Tr}\,L\hat{\xi} \\
&\qquad = \frac{\xi_1-\xi_2}{2a_+}\,\mathrm{Tr}\,((U_{-n}P_k)^* L\,(U_{-n}P_k)L - a_+^2 I_2) = 2a_+(\xi_2-\xi_1)\cdot|(U_{-n}P_k)_{12}|^2,
\end{aligned}$$

thus

$$F(0,0,V) - F(0,0,I) = \frac{2ia_+(\xi_2-\xi_1)}{\rho(\lambda_0)} \sum_{k=-n+1}^{n} \big(|(U_{-n}P_k)_{12}|^2 - |(U_{-n})_{12}|^2\big)\,. \tag{3.49}$$

Hence, we have to study

$$\Phi_{k_1,\dots,k_p}(\overline{a},\overline{b}) = \Big\langle \prod_{j=1}^{p} \big(|(U_{-n}P_{k_j})_{12}|^2 - |(U_{-n})_{12}|^2\big) \Big\rangle_{\widetilde{\mu}}, \tag{3.50}$$

where

$$\langle\dots\rangle_{\widetilde{\mu}} = I_{\widetilde{\mu}}^{-1}\int(\dots)d\widetilde{\mu}(V,\widetilde{A}). \tag{3.51}$$

Let $p < Cn/W$ for some constant C. Introduce i.i.d $\{t_j\}$ such that the density of the distribution has the form

$$\rho(t_j) = \frac{(a_+-a_-)^2}{2}\, t_j \exp\{-t_j^2(a_+-a_-)^2\}\cdot \mathbf{1}_{0<t_j<W/2} \tag{3.52}$$

and consider the unitary matrices

$$\widetilde{V}_j = \begin{pmatrix} \widetilde{r}_j e^{i\widetilde{\theta}_j} & \widetilde{v}_j e^{i\theta_j} \\ -\widetilde{v}_j e^{-i\theta_j} & \widetilde{r}_j e^{-i\widetilde{\theta}_j} \end{pmatrix}, \tag{3.53}$$

where

$$\begin{aligned}
\widetilde{v}_j &= \frac{t_j}{W}\cdot\Big(1+\frac{\widetilde{a}_j-\widetilde{b}_j}{W(a_+-a_-)}\Big)^{-1/2}\Big(1+\frac{\widetilde{a}_{j-1}-\widetilde{b}_{j-1}}{W(a_+-a_-)}\Big)^{-1/2},\\
\widetilde{r}_j &= (1-\widetilde{v}_j^2)^{1/2},
\end{aligned}$$

and $\theta_j, \widetilde{\theta}_j \in [-\pi,\pi)$.

It follows from the properties of the Haar measure on the unitary group and and (3.46) that

$$\begin{aligned}
\widetilde{\Phi}_{k_1,\dots,k_p}(\overline{a},\overline{b}) &:= \Big\langle \prod_{j=1}^{p}\Big(|(U_{-n}\prod_{l=-n+1}^{k_j}\widetilde{V}_l)_{12}|^2 - |(U_{-n})_{12}|^2\Big)\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j} \\
&= \Phi_{k_1,\dots,k_p}(\overline{a},\overline{b}) + O(e^{-cW^2}),
\end{aligned} \tag{3.54}$$

where $\langle\dots\rangle_{t_j,\theta_j,\widetilde{\theta}_j}$ means the expectation over $\{t_j\}$ with respect to the measure with the distribution (3.52) and over $\{\theta_j\}$,$\{\widetilde{\theta}_j\}$ from $-\pi$ to π.

Denote

$$s_j = 1 - \Big(1 + \frac{\widetilde{a}_j - \widetilde{b}_j}{W(a_+ - a_-)}\Big)\Big(1 + \frac{\widetilde{a}_{j-1} - \widetilde{b}_{j-1}}{W(a_+ - a_-)}\Big). \tag{3.55}$$

Expanding $\widetilde{V}_j$ with respect to s_j we get

$$\widetilde{V}_j = \widetilde{V}_j(0) + \frac{t_j}{W}((1 - s_j)^{-1/2} - 1)V_j^1 + \frac{t_j^2}{W^2}\sum_{r=1}^{\infty} V_j^{(r)} s_j^r,$$

where $\widetilde{V}_j(0)$ is a unitary matrix (and hence $\|\widetilde{V}_j(0)\| \le 1$),

$$\widetilde{V}_j^1 = \begin{pmatrix} 0 & e^{i\theta_j} \\ -e^{-i\theta_j} & 0 \end{pmatrix}, \quad \|\widetilde{V}_j^{(r)}\| \le C^r \quad (r = 1, 2, \ldots),$$

and $\{\widetilde{V}_j^{(r)}\}$ are diagonal matrices.

Since the integrals of $e^{im\theta_j}$ equal 0 for $m \ne 0$ and 2π for $m = 0$, we conclude that if we replace the coefficient in front of $e^{i\theta_j}$ and $e^{-i\theta_j}$ by the bounds for their absolute values, then after the averaging with respect to θ_j the resulting coefficients in front of s_j^k will grow. Hence

$$\widetilde{\Phi}_{k_1,\ldots,k_p}(\overline{a}, \overline{b}) - \widetilde{\Phi}_{k_1,\ldots,k_p}(0,0) \prec \Big\langle \prod \Big|1 + \frac{t_j}{W}\, e^{i\theta_j} s_j g(s_j) + \frac{t_j^2}{W^2} s_j g(s_j)\Big|^{2p} \Big\rangle_{t_j,\theta_j} - 1, \tag{3.56}$$

where $g(t) = C_0/(1 - Ct)$ with some n-independent C, C_0. Moreover,

$$\Big\langle \frac{t_j^{2k}}{W^{2k}} \Big\rangle_{t_j} \le \frac{k!}{(a_+ - a_-)^{2k} W^{2k}},$$

and thus we conclude

$$\Big\langle \prod \Big|1 + \frac{t_j}{W}\, e^{i\theta_j} s_j g(s_j) + \frac{t_j^2}{W^2} s_j g(s_j)\Big|^{2p} \Big\rangle_{t_j,\theta_j} \prec \prod \Big(1 + \frac{2p}{W^2} s_j g(s_j) + \frac{p^2}{W^2} s_j^2 g(s_j)^2\Big)$$

Since for $p \le Cn/W$ we have $2p/W^2 \le W^{-1}$, $p^2/W^2 = o(1)$, Remark 1 yields

$$\begin{aligned}
&\Big|\Big\langle (\widetilde{\Phi}_{k_1,\ldots,k_p}(\overline{a}, \overline{b}) - \widetilde{\Phi}_{k_1,\ldots,k_p}(0,0)) \cdot \Pi_1 \cdot \Pi_2 \Big\rangle_0\Big| \\
&\le \Big\langle \Big(\prod \Big(1 + \frac{2p}{W^2} s_j g(s_j) + \frac{p^2}{W^2} s_j^2 g(s_j)^2\Big) - 1\Big) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \Big\rangle_{0,*} + e^{-C\delta^2 W} \\
&\le \Big\langle \Big(\exp\Big\{\sum_{i=-n}^{n} \Big(\frac{Cp}{W^2} \cdot \frac{\widetilde{a}_i + \widetilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\widetilde{a}_i^2 + \widetilde{b}_i^2}{W^2}\Big)\Big\} - 1\Big) \cdot \Pi_{1,*} \cdot \Pi_{2,*} \Big\rangle_{0,*} + e^{-C\delta^2 W} \\
&\le \Big\langle \Big(\exp\Big\{\sum_{i=-n}^{n} \Big(\frac{Cp}{W^2} \cdot \frac{\widetilde{a}_i + \widetilde{b}_i}{W} + \frac{p^2 c}{W^2} \cdot \frac{\widetilde{a}_i^2 + \widetilde{b}_i^2}{W^2}\Big)\Big\} - 1\Big)^2 \Big\rangle_{0,*}^{1/2} \cdot \Big\langle \Pi_{1,*}^2 \cdot \Pi_{2,*}^2 \Big\rangle_{0,*}^{1/2} + e^{-C\delta^2 W},
\end{aligned}$$

where Π_1, Π_2 are the products of the Taylor's series for $\exp\{\varphi_+(\widetilde{a}_j/W)\}$ and for $\exp\{\varphi_-(\widetilde{b}_j/W)\}$, and $\Pi_{1,*}$, $\Pi_{2,*}$ are obtained form Π_1, Π_2 by changing the coefficients to their absolute values.

We proved before that the second factor is $1+o(1)$. Moreover, taking the gaussian integral from the first factor (similarly to Remark 1.1) we obtain that

$$\Big|\Big\langle(\widetilde{\Phi}_{k_1,\dots,k_p}(\overline{a},\overline{b})-\widetilde{\Phi}_{k_1,\dots,k_p}(0,0))\cdot\Pi_1\cdot\Pi_2\Big\rangle_0\Big|\le\exp\{\frac{cp^2n}{W^3}\}-1\le\exp\{\frac{cpn^2}{W^4}\}-1,$$

and thus, since $p<Cn/W$,

$$\sum_{p=1}^{Cn/W}\frac{1}{p!}\sum_{k_1,\dots,k_p}n^{-p}\Big|\Big\langle(\widetilde{\Phi}_{k_1,\dots,k_p}(\overline{a},\overline{b})-\widetilde{\Phi}_{k_1,\dots,k_p}(0,0))\cdot\Pi_1\cdot\Pi_2\Big\rangle_0\Big|\le\exp\{e^{Cn^2/W^4}\}-e=o(1).$$

If $p\gg n/W$, then $1/p!\ll e^{-Cn/W}$ and hence we can change $\langle\dots\rangle_0$ to $\langle\dots\rangle_{0,*}$ (see Lemma 2).

We are left to prove that

$$\widetilde{\Phi}_{k_1,\dots,k_p}(0,0)=\Big\langle\prod_{j=1}^{p}\Big(|(U_{-n})_{12}\widetilde{P}_{k_j}(0)|^2-|(U_{-n})_{12}|^2\Big)\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}=o(1),$$

where

$$\widetilde{P}_{k_j}(0)=\prod_{l=-n+1}^{k_j}\widetilde{V}_l(0).$$

To this end we write

$$\begin{aligned}&\Big\langle\prod_{j=1}^{p}\Big||(U_{-n}\widetilde{P}_{k_j}(0))_{12}|^2-|(U_{-n})_{12}|^2\Big|\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\le\Big\langle\Big||(U_{-n}\widetilde{P}_{k_1}(0))_{12}|^2-|(U_{-n})_{12}|^2\Big|\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\\&\le\Big\langle\Big||(U_{-n}\widetilde{P}_{k_1-1}(0))_{12}(\widetilde{V}_{k_1}(0))_{22}+(U_{-n}\widetilde{P}_{k_1-1}(0))_{11}(\widetilde{V}_{k_1}(0))_{12}|^2-|(U_{-n})_{12}|^2\Big|\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\\&=\Big\langle|(U_{-n}\widetilde{P}_{k_1-1}(0))_{11}|^2\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\cdot\Big\langle|(\widetilde{V}_{k_1}(0))_{12}|^2\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\\&+\Big\langle\Big||(U_{-n}\widetilde{P}_{k_1-1}(0))_{12}|^2-|(U_{-n})_{12}|^2\Big|\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\\&\le\frac{C}{W^2}+\Big\langle\Big||(U_{-n}\widetilde{P}_{k_1-1}(0))_{12}|^2-|(U_{-n})_{12}|^2\Big|\Big\rangle_{t_j,\theta_j,\widetilde{\theta}_j}\le\dots\le\frac{Cn}{W^2}=o(1).\end{aligned}$$

Thus, the main term is one without $|(V_j)_{12}|^2$, i.e. we can substitute $V_j=1$, $j=-n+1,\dots,n$ in (3.21).

Hence, integrating over $\{V_j\}_{j=-n+1}^{n}$ we obtain

$$\begin{aligned}\Sigma_\pm=&W^{-8n-2}2^{2n}e^{2(2n+1)c_0}e^{i\pi(\xi_1-\xi_2)}\int\limits_{U(2)}\int\limits_{|\widetilde{a}_j|,|\widetilde{b}_j|\le W^{1-\kappa}}\mu_{c_+}(a)\,\mu_{c_-}(b)\\&\times\exp\Big\{-\sum_{j=-n}^{n}\varphi_+(\widetilde{a}_j/W)-\sum_{j=-n}^{n}\varphi_-(\widetilde{b}_j/W)\Big\}\\&\times e^{\frac{i(2n+1)}{n\rho(\lambda_0)}\,(\mathrm{Tr}\,U_{-n}^*LU_{-n}\hat{\xi}-\mathrm{Tr}\,L\hat{\xi})}(a_+-a_-+(\tilde{a}_{-n}-\tilde{b}_{-n})/W)\\&\times(a_+-a_-+(\tilde{a}_n-\tilde{b}_n)/W)d\,\mu(U_{-n})\prod_{q=-n}^{n}d\tilde{a}_q\,d\tilde{b}_q(1+o(1))\end{aligned}\tag{3.57}$$

Integrating over U_{-n} by the Itsykson-Zuber formula and using step 1 we get finally

$$\begin{aligned}\Sigma_\pm =&\frac{W^{-8n-2}2^{2n}e^{2(2n+1)c_0}\big(e^{i\pi(\xi_1-\xi_2)}-e^{i\pi(\xi_2-\xi_1)}\big)}{4i\pi(\xi_1-\xi_2)}\int\limits_{|\tilde a_j|,|\tilde b_j|\le W^{1-\kappa}}\prod_{q=-n}^{n}d\tilde a_q\,d\tilde b_q\cdot\mu_{c_+}(a)\,\mu_{c_-}(b)\\ &\times(a_+-a_-+(\tilde a_{-n}-\tilde b_{-n})/W)(a_+-a_-+(\tilde a_n-\tilde b_n)/W)(1+o(1)) \qquad (3.58)\\ &=\frac{\pi e^{2(2n+1)c_0}\rho(\lambda_0)^2(4\pi)^{2n+1}\sin(\pi(\xi_1-\xi_2))}{W^{8n+2}(\xi_1-\xi_2)}\Big|\det{}^{-1}\Big(-\Delta+\frac{2c_+}{W^2}\Big)\Big|(1+o(1))\end{aligned}$$

By the same way we have

$$\Sigma_\mp =\frac{\pi e^{2(2n+1)c_0}\rho(\lambda_0)^2(4\pi)^{2n+1}\sin(\pi(\xi_2-\xi_1))}{W^{8n+2}(\xi_2-\xi_1)}\Big|\det{}^{-1}\Big(-\Delta+\frac{2c_+}{W^2}\Big)\Big|, \qquad (3.59)$$

and thus (since $\overline{c}_+=c_-$)

$$\Sigma_\pm+\Sigma_\mp =\frac{e^{2(2n+1)c_0}\rho(\lambda_0)^2(4\pi)^{2n+1}\pi^2}{W^{8n+2}}\cdot\frac{\sin\pi(\xi_1-\xi_2)}{\pi(\xi_1-\xi_2)}\cdot\Big|\det{}^{-1}\Big(-\Delta+\frac{2c_+}{W^2}\Big)\Big|. \qquad (3.60)$$

3.3.2 Σ_+ and Σ_-.

Similarly to (3.21) we get

$$\begin{aligned}\Sigma_+ =&W^{-8n-4}2^{2n}e^{2(2n+1)r_++i\pi(\xi_1+\xi_2)}\int\limits_{|\tilde a_j|,|\tilde b_j|\le W^{1-\kappa}}\int\limits_{U(2)}\mu_{c_+}(a)\mu_{c_+}(b)\\ &\times e^{\sum_{j=-n+1}^{n}\mathrm{Tr}\,\big(V_j^*\tilde A_jV_j\tilde A_{j-1}-\tilde A_j\tilde A_{j-1}\big)} \qquad (3.61)\\ &\times e^{-\sum_{k=-n}^{n}(\varphi_+(\tilde a_k/W)+\varphi_+(\tilde b_k/W))+\frac{i}{n\rho(\lambda_0)}\sum_{p=-n}^{n}\mathrm{Tr}\,(U_{-n}P_p)^*(\tilde A_p/W)\,(U_{-n}P_p)\hat\xi}\\ &\times\prod_{l=-n}^{n}(\tilde a_l-\tilde b_l)^2d\mu(U_{-n})\prod_{q=-n+1}^{n}d\mu(V_q)\,d\overline{a}\,d\overline{b}(1+o(1)).\end{aligned}$$

Since

$$\exp\Big\{\frac{i}{n\rho(\lambda_0)}\sum_{p=-n}^{n}\mathrm{Tr}\,(U_{-n}P_p)^*(\tilde A_p/W)\,(U_{-n}P_p)\hat\xi\Big\}\prec\exp\Big\{\sum_{p=-n}^{n}C(\tilde a_p+\tilde b_p)/nW\Big\},$$

by the same argument as for $\Sigma_\pm$ we get

$$\begin{aligned}\Sigma_+ =&2^{2n}W^{-4(2n+1)}e^{2(2n+1)r_++i\pi(\xi_1+\xi_2)}\int\limits_{|\tilde a_j|,|\tilde b_j|\le W^{1-\kappa}}\prod_{q=-n}^{n}d\tilde a_q\,d\tilde b_q\\ &\times\mu_{c_+}(a)\mu_{c_+}(b)\,(\tilde a_{-n}-\tilde b_{-n})\,(\tilde a_n-\tilde b_n)\,(1+o(1)) \qquad (3.62)\\ =&2^{2n}W^{-4(2n+1)}e^{2(2n+1)r_++i\pi(\xi_1+\xi_2)}\int\limits_{\mathbb{R}}\prod_{q=-n}^{n}d\tilde a_q\,d\tilde b_q\\ &\times\mu_{c_+}(a)\mu_{c_+}(b)\,(\tilde a_{-n}-\tilde b_{-n})\,(\tilde a_n-\tilde b_n)\,(1+o(1))\\ =&2^{2n}W^{-4(2n+1)}e^{2(2n+1)r_++i\pi(\xi_1+\xi_2)}D^{-1}_{-n,n}\det{}^{-1/2}D,\end{aligned}$$

where

$$D = -\Delta + \frac{2c_+}{W^2}.$$

It is easy to see (see the proof of Lemma 2) that

$$|D^{-1}_{-n,n}| = 1/\det D = o(1).$$

Hence, since $\Re r_+ = c_0$, we get

$$|\Sigma_+| \le 2^{2n} W^{-8n-4} e^{2(2n+1)c_0 + i\pi(\xi_1+\xi_2)} |\det{}^{-1} D|$$

the order of Σ_+ is smaller than the order of $\Sigma_\pm$. Thus we can write

$$\Sigma = \frac{e^{2(2n+1)c_0}\rho(\lambda_0)^2(4\pi)^{2n+1}\pi^2}{W^{8n+2}} \cdot \frac{\sin\pi(\xi_1-\xi_2)}{\pi(\xi_1-\xi_2)} \cdot \Big|\det{}^{-1}\Big(-\Delta + \frac{2c_+}{W^2}\Big)\Big|(1+o(1)). \quad (3.63)$$

This, (3.3) and (2.14) yield Theorem 1.

4 Appendix

Proof of Lemma 1 Note that

$$\begin{aligned} f_*(a_\pm) &= 0, \\ \frac{d}{dx} f_*(x)\Big|_{x=a_\pm} &= \Big(x - \frac{x}{x^2+\lambda_0^2/4}\Big)\Big|_{x=a_\pm} = 0, \\ \frac{d^2}{dx^2} f_*(x)\Big|_{x=a_\pm} &= \Big(1 - \frac{1}{x^2+\lambda_0^2/4} + \frac{2x^2}{(x^2+\lambda_0^2/4)^2}\Big)\Big|_{x=a_\pm} = 2(1-\lambda_0^2/4). \end{aligned}$$

Thus function $f_*(x)$ attains its minimum at $a_\pm$ and expanding $f_*(x)$ in $x \in (a_\pm - \delta, a_\pm + \delta)$ we get

$$f_*(x) = (1-\lambda_0^2/4)(x-a_\pm)^2 + O(\delta^3). \quad (4.1)$$

This yields (3.13). Besides, it is easy to see, that if we take $\gamma = \frac{1}{2}(1-\lambda_0^2/4)$ then we obtain (3.14) for some sufficiently small δ. $\square$

Proof of Lemma 2

1) Set

$$-\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 1 \end{pmatrix}, \quad -\Delta_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 1 \end{pmatrix}. \quad (4.2)$$

Define

$$T_n(x) = \det(-\Delta_1 + x\cdot I), \quad S_n(x) = \det(-\Delta + x \cdot I). \quad (4.3)$$

It is easy to check that

$$T_n(x) = (2+x)T_{n-1}(x) - T_{n-2}(x), \quad T_1(x) = 1+x, \quad T_2(x) = x^2+3x+1, \tag{4.4}$$
$$S_n(x) = (1+x)T_{n-1}(x) - T_{n-2}(x). \tag{4.5}$$

Solving the recurrent relation (4.4) we get

$$T_m(x) = \frac{\zeta^{m+1}+\zeta^{-m}}{\zeta+1}, \quad S_m(x) = \frac{(\zeta^m-\zeta^{-m})(\zeta-1)}{\zeta+1} \tag{4.6}$$

where

$$\zeta = \frac{2+x+\sqrt{x^2+4x}}{2}.$$

For $x = 2\gamma/W^2$

$$\zeta = 1+\sqrt{2\gamma}/W+\gamma/W^2+O(W^{-3}), \quad W\to\infty.$$

This and (4.5) – (4.6) yield

$$T_m(2\gamma/W^2) = \cosh\frac{m\sqrt{2\gamma}}{W}(1+o(1)), \quad S_m(2\gamma/W^2) = \frac{\sqrt{2\gamma}}{W}\sinh\frac{m\sqrt{2\gamma}}{W}(1+o(1)),$$

and thus (3.16). Also it is easy to see that

$$G_{ii}^{(m)}(\gamma) = \frac{T_{i-1}(2\gamma/W^2)T_{m-i}(2\gamma/W^2)}{S_m(2\gamma/W^2)} \le \frac{C_\gamma W}{\sqrt{2\gamma}}\tanh^{-1}\frac{m\sqrt{2\gamma}}{W}(1+o(1)).$$

Moreover,

$$\begin{aligned} G_{11}^{(m)}(\gamma) - G_{1m}^{(m)}(\gamma) = \frac{T_{m-1}(2\gamma/W^2)-1}{S_m(2\gamma/W^2)} \\ = C_\gamma W\tanh\frac{m\sqrt{2\gamma}}{2W}(1+o(1)) \le C_\gamma^1\min\{m,W\}. \end{aligned}$$

2) Take $m \ge CW$ and $\alpha\in\mathbb{R}, \alpha>0$. Note that for any sufficiently small $\delta>0$ and $\varepsilon>0$

$$\begin{aligned} Z_\alpha^{(m)} - Z_{\delta,\alpha}^{(m)} &= \int\limits_{\max|x_i|>\delta W} e^{-\frac{1}{2}\sum\limits_{j=2}^{m}(x_j-x_{j-1})^2-\frac{\alpha}{W^2}\sum\limits_{j=1}^{m}x_j^2}\prod_{q=1}^{m}dx_q \\ &\le \sum_{i=1}^{m}\int e^{\frac{\varepsilon^2}{2}(x_i^2-W^2\delta^2)-\frac{1}{2}\sum\limits_{j=2}^{m}(x_j-x_{j-1})^2-\frac{\alpha}{W^2}\sum\limits_{j=1}^{m}x_j^2}\prod_{q=1}^{m}dx_q \\ &= \sum_{i=1}^{m}\frac{e^{-\varepsilon^2\delta^2W^2/2}}{\sqrt{2\pi}}\int dt\, e^{-t^2/2}\int\prod_{q=1}^{m}dx_q\, e^{\varepsilon t x_i-\frac{1}{2}\sum\limits_{j=2}^{m}(x_j-x_{j-1})^2-\frac{\alpha}{W^2}\sum\limits_{j=1}^{m}x_j^2} \\ &= \frac{m\, e^{-\varepsilon^2\delta^2W^2/2}}{\sqrt{2\pi}}\cdot Z_\alpha^{(m)}\cdot\sum_{i=1}^{m}\int e^{-t^2/2+\varepsilon^2G_{ii}^{(m)}(\alpha)t^2/2}dt, \end{aligned} \tag{4.7}$$

where $G^{(m)}$ is defined in (3.17).

Let us take $\varepsilon^2 = (G_{ii}^{(m)}(\alpha))^{-1}/2$ in (4.7). Then taking into account (3.18) and $CW \le m \le 2n+1$, we obtain for $\alpha \in \mathbb{R}, \alpha > 0$

$$\frac{Z_\alpha^{(m)} - Z_{\delta,\alpha}^{(m)}}{Z_\alpha^{(m)}} \le C_1\, e^{-C_2\delta^2 W}. \tag{4.8}$$

Since $m \le 2n+1$, according to the first assertion of the lemma we get

$$\frac{|Z_{\gamma_1}^{(m)}|}{|Z_{\gamma_2}^{(m)}|} = (1 + C/W)^m \le e^{C_1 n/W}, \quad m, W \to \infty. \tag{4.9}$$

This and (4.8) yield for $m \ge CW$, $\gamma \in \mathbb{C}$, $\Re\gamma > 0$

$$\frac{|Z_\gamma^{(m)} - Z_{\delta,\gamma}^{(m)}|}{|Z_\gamma^{(m)}|} \le \frac{Z_{\Re\gamma}^{(m)} - Z_{\delta,\Re\gamma}^{(m)}}{Z_{\Re\gamma}^{(m)}} \cdot \frac{Z_{\Re\gamma}^{(m)}}{|Z_\gamma^{(m)}|} \le C_1\, e^{-C_2\delta^2 W + Cn/W} \le C_1\, e^{-C_3\delta^2 W}.$$

Since $W^2 = n^{1+\theta}$ we can take $\delta = W^{-\kappa}$ with $\kappa < \theta/(1+\theta)$.

Take now any m. Using the assertion (1) of the Lemma we can write for any $\varepsilon > 0$

$$\begin{aligned}
&(Z_\alpha^{(m)})^{-1} \int\limits_{x_k - x_1 > \delta W} e^{-\frac{1}{2}\sum\limits_{j=2}^{m}(x_j - x_{j-1})^2 - \frac{\alpha}{2W^2}\sum\limits_{j=1}^{m} x_j^2} \prod_{q=1}^{m} dx_q \\
&\le (Z_\alpha^{(m)})^{-1} \int e^{\varepsilon(x_k - x_1 - \delta W) - \frac{1}{2}\sum\limits_{j=2}^{k}(x_j - x_{j-1})^2 - \frac{\alpha}{2W^2}\sum\limits_{j=1}^{k} x_j^2} \prod_{q=1}^{k} dx_q \\
&\times \int e^{-\frac{1}{2}\sum\limits_{j=k+2}^{m}(x_j - x_{j-1})^2 - \frac{\alpha}{2W^2}\sum\limits_{j=k+1}^{m} x_j^2} \prod_{q=k+1}^{m} dx_q \\
&\le \frac{Z_\alpha^{(k)} Z_\alpha^{(m-k)}}{Z_\alpha^{(m)}} \cdot e^{-\varepsilon\delta W + c\varepsilon^2 (G_{11}^{(k)} - G_{1k}^{(k)})} \le W e^{-\varepsilon\delta W + C\varepsilon^2 \min\{m, W\}} \le e^{-C_1\delta^2 W}.
\end{aligned}$$

3) It is easy to see that

$$-\frac{\alpha x^2}{2} + k_i \log|x| \le -\frac{\alpha x^2}{4} + \frac{k_i}{2}\log\frac{2k_i}{\alpha}.$$

Thus, using assertions (1) – (2) of the Lemma and (4.9) we obtain

$$\begin{aligned}
&|Z_\gamma^{(m)}|^{-1} \Big| \int\limits_{\max|x_i| > \delta W} \prod_{j\in S} x_j^{k_j} \cdot \mu_\gamma^{(m)}(x) \prod_{q=1}^{m} dx_q \Big| \\
&\le |Z_\gamma^{(m)}|^{-1} e^{\sum\limits_{i=1}^{s} \frac{k_i}{2}\log\frac{2k_i}{\Re\gamma}} \cdot \int\limits_{\max|x_i| > \delta W} \mu_{\Re\gamma/2}^{(m)}(x) \prod_{q=1}^{m} dx_q \\
&\le e^{C_1 k \log k + C_2 m/W} \frac{|Z_{\Re\gamma/2}^{(m)} - Z_{\delta,\Re\gamma/2}^{(m)}|}{|Z_{\Re\gamma/2}^{(m)}|} \le e^{-CW\delta^2},
\end{aligned}$$

where the last inequality holds since $k \le Cm/W \ll W$. $\square$

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